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RESEARCH

The Fractional Laplacian with Reflections

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Abstract

Motivated by the notion of isotropic α-stable Lévy processes confined, by reflections, to a bounded open Lipschitz set $D \subset \mathbb{R}^d$, we study some related analytical objects. Thus, we construct the corresponding transition semigroup, identify its generator and prove exponential speed of convergence of the semigroup to a unique stationary distribution for large time.

Keywords Fractional Laplacian · Reflection · Stationary distribution

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1 Introduction

1.1 Setting, Goals, and Results

Consider the isotropic α-stable Lévy process $(Y_t, t \geq 0)$ in $\mathbb{R}^d$. The intensity of jumps for this process is given by $\nu(x, \mathrm{d}y) = c_{d,\alpha}|x-y|^{-d-\alpha}\mathrm{d}y$, which is the integro-differential kernel of the fractional Laplacian on $\mathbb{R}^d$ (for details, see below). Given an open set $D \subset \mathbb{R}^d$, our interest lies in a Markov process $(X_t, t \geq 0)$ that coincides with Y as long as Y remains within D. However, at the time τ_D of the first exit of Y from D, we intend to perform a *reflection*: instead of allowing X to leave D by going to the point $z := Y_{\tau_D} \in D^c$, our process X should be *restarted* immediately (at time τ_D) at a point $y \in D$, without venturing into D^c. The point y is chosen (randomly) according to a probability measure $\mu(z, \mathrm{d}y)$, which

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depends on z, otherwise independently of prior events. Based on this heuristic description, we expect the intensity of jumps of $(X_t, t \geq 0)$ to be the following integral kernel on D,

$$\gamma(x, \mathrm{d}y) := \nu(x, \mathrm{d}y) + \int_{D^c} \nu(x, \mathrm{d}z)\mu(z, \mathrm{d}y). \tag{1.1}$$

Actually, the outcome of this work is not a Markov process, but a *conservative Markovian semigroup* $(K(t), t > 0)$ having γ as the kernel of the generator. We also prove that $(K(t), t > 0)$ has a unique stationary density and is *exponentially asymptotically stable*. Regarding the construction and analysis of the process X, we will cover them in [17]. Now, let us state our assumptions on D and μ:

Hypothesis 1.1 Let $D \neq \emptyset$ be an open bounded Lipschitz subset of $\mathbb{R}^d$ and let $\mu : D^c \times \mathscr{B}(D) \to [0, 1]$ have the following properties:

(i) For every $A \in \mathscr{B}(D)$, the map $z \mapsto \mu(z, A)$ is Borel measurable, and for every $z \in D^c$, $\mu(z, \cdot)$ is a Borel probability measure on D.
(ii) There exists a compact set $H \subset D$ such that $\vartheta := \inf_{z \in D^c} \mu(z, H) > 0$.
(iii) The map $z \mapsto \mu(z, \cdot)$ is weakly continuous at ∂D; that is, if points $z_n \in D^c$ converge to $z \in \partial D$, then the measures $\mu(z_n, \cdot)$ converge weakly to $\mu(z, \cdot)$.

Hypothesis 1.1 is assumed throughout, though we recall it from time to time. Let us comment on the assumptions. The geometric regularity of D is technically important but can be relaxed; see Remark 3.6. Condition (i) means, in short, that the *repulsion kernel* μ is a *stochastic kernel*. The lower bound (ii) is crucial for Theorem 3.4 and, informally, for avoiding an infinite number of reflections within finite time. The weak continuity (iii) allows for a description of the corresponding infinitesimal generator and boundary conditions in terms of continuous functions on $\overline{D}$; see Theorem 5.4 and Remark 5.5.

1.2 Context and Literature

Reflections similar to the ones that we study for jump processes appeared first in the work of Feller [31] concerning one-dimensional diffusions. Feller coined the term *instantaneous return processes* for the resulting Markov processes. On the level of the corresponding generator of the process, which is a second order elliptic differential operator in [31], these reflections lead to certain nonlocal boundary conditions. Further operators of this form, including those in higher dimension, have been explored by various techniques and authors, including Galakhov and Skubachevskiĭ [33], Ben-Ari and Pinski [4], Arendt, Kunkel, and Kunze [1], and Kunze [47]; see also the monograph of Taira [64] and the references therein.

Boundary conditions for jump processes and nonlocal operators remain an open subject, although various mechanisms of reflection from D^c and Neumann-type conditions have been proposed in the literature. When reflecting jump processes from D^c to D, one faces the choice of making X_{τ_D} depend on $Y_{\tau_D -}$, on Y_{τ_D}, or both. These scenarios are notably richer than those for diffusions. Bogdan, Burdzy, and Chen [9] propose both censored and actively reflected processes, with the reflection depending (deterministically) solely on $Y_{\tau_D -}$. For D being the half-space, Barles, Chasseigne, Georgelin, and Jakobsen [3] discuss geometrically motivated reflections that deterministically depend on $(Y_{\tau_D -}, Y_{\tau_D})$. Dipierro, Ros-Oton, and Valdinoci [28, p. 378] postulate a random mechanism of reflection that employs $\mu(z, \mathrm{d}y) = \nu(z, \mathrm{d}y)/\nu(z, D)$. Notably, the papers [3, 28] discuss Neumann-type problems, but neither the semigroup nor the corresponding Markov process, both of which are highly nontrivial to construct and study [9]. In comparison, Vondraček [65] constructs a Markov process X that returns to D following the distribution $\nu(Y_{\tau_D}, \mathrm{d}y)/\nu(Y_{\tau_D}, D)$, but the process X remains at $Y_{\tau_D} \in D^c$ for a unit exponential time before returning to D. This arrangement helps avoid the problematic scenario of an infinite number of passes between D and D^c in finite time, while also taking [65] beyond the tentative setting of instantaneous reflections in [3, 28], and our paper.

Reflecting Markov processes may be considered as an instance of *stochastic resetting*, which is a hot topic in statistical physics concerned, among others, with equilibrium distributions, search optimization, renewal theory and modelling; see, e.g., Evans, Majumdar, and Schehr [30], Garbaczewski and Żaba [34], and Stanislavsky and Weron [63]. Specific reflections, *resurrections*, or *recurrent extensions* of stochastic processes with *scaling* are studied by Kim, Song, and Vondraček [40] for the halfline using the Lamperti transform. See also Rivero [56] and Fitzsimmons [32]. Kim, Song, and Vondraček [41] explore half-spaces using Dirichlet forms; see also Chen and Song [23]. Recent results for the entire real line can be found in Pantí, Pardo, and Rivero [53] and Iksanov and Pilipenko [37]; see also Chaumont, Pantí, and Rivero [21]. Furthermore, Bobrowski [6] studies resurrections or *concatenations* on metric graphs via *exit laws* and resolvents. We should add that a general probabilistic approach to concatenation or *piecing-out* of Markov processes was proposed by Ikeda, Nagasawa, and Watanabe [36], see also Meyer [52]. Sharpe [62] provides an introduction and further references to the method. For a recent presentation and references to concatenation of *right processes*, we refer to Werner [66].

1.3 Approach, Perspectives, and Content

We construct the Markovian transition semigroup corresponding to the kernel γ by the method of nonlocal Schrödinger perturbations of integral kernels by Bogdan and Sydor [19].

In the special case of transition kernels, the method may handle analytic aspects of concatenation of Markov processes. Namely, given a sub-Markovian kernel, a suitable repulsion kernel defines a larger transition kernel, possibly Markovian, which corresponds to a specific partial differential equation with (nonlocal) boundary conditions. For instance, see Corollary 5.6.

In the present setting we consider a class of repulsion kernels μ for the Dirichlet heat kernel of the fractional Laplacian. On the one hand, the restriction to the fractional Laplacian is dictated merely by the technical convenience and general interest in this operator, so generalizations are quite obvious. On the other hand, the papers [3, 9, 28, 40, 41, 65] make do without the lower bound for the repulsion kernel that we assume in Hypothesis 1.1 (ii). Therefore the above references and the present paper should be considered as different ramifications of the problem of constructing operators, semigroups and Markov processes with specific boundary conditions. As we already mentioned, this area of research is motivated by the Neumann-type boundary-value problems and the problem of piecing-out or concatenation of Markov processes. Its objective, beyond the construction, are the questions of the large-time and boundary behavior of the resulting semigroup and process, as well as applications to nonlocal differential equations with those boundary conditions.

The paper is organized as follows. Section 2 provides essential preliminaries, including an introduction to the fractional Laplacian and related potential theory. It also introduces the Dirichlet heat kernel $(p_D(t), t > 0)$ of the set D. In Section 3, we proceed to construct the kernel $(k(t), t > 0)$ that defines the semigroup $(K(t))$. We also prove that $\int_D k(t, x, y)\mathrm{d}y = 1$ for all $x \in D$ and $t > 0$. Section 4 delves into the study of the resolvent of $(K(t))$. In Section 5, we *characterize* the generator of the semigroup and discuss the associated boundary conditions. Additionally, we provide a solution to a typical boundary value problem. In Section 6 we prove the existence of a unique invariant measure (density) and the exponential convergence of the semigroup to the stationary measure for large time. The reader may also find a few examples illustrating our results. Example 3.7 and Remark 3.8 (restarting at a fixed point $x_0 \in D$) show that, in general, the semigroup $(K(t))$ is neither symmetric nor does it act on L^2. Examples 4.6, 4.7, and 4.8 focus on the boundary conditions induced by μ. In Remark 6.3, we propose directions for future research and generalizations.

As we mentioned, the construction of the semigroup $(K(t))$ is purely analytic, based on nonlocal Schrödinger perturbation, or Duhamel formula applied to p^D. The reader interested in the existence and properties of a Markov process resulting from $(K(t))$ is referred to the forthcoming paper [17] by the authors.

2 Preliminaries

We often use := for definitions, e.g., $\mathbb{N}_0 := \{0, 1, 2 \ldots\}$ and $\mathbb{N} := \{1, 2, \ldots\}$. $\mathbb{1}_A$ denotes the indicator function of A and $\mathbb{1}$ is the indicator of the full space. We let $d \in \mathbb{N}$ and consider the Euclidean space $\mathbb{R}^d$. All the sets, functions, measures and kernels considered in the paper are Borel. If not stated otherwise, functions take values in the extended real line. For $x \in \mathbb{R}^d$ and $r \in (0, \infty)$ we denote by $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$ the ball with radius r and center at x. We require integrals to be nonnegative or absolutely convergent.

2.1 Fractional Laplacian

Let $\alpha \in (0, 2)$, and

$$\nu(x) := c_{d,\alpha}|x|^{-d-\alpha}, \quad x \in \mathbb{R}^d,$$

where

$$c_{d,\alpha} := \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}.$$

The constant $c_{d,\alpha}$ is chosen in such a way that

$$|\xi|^\alpha = \int_{\mathbb{R}^d} (1 - \cos \xi \cdot x)\nu(x)\mathrm{d}x\,, \quad \xi \in \mathbb{R}^d.$$

According to Fourier inversion and the Lévy–Khinchine formula, there is a convolution semigroup of smooth probability densities $(p(t), t > 0)$ such that

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\xi\cdot x} p(t, x)\mathrm{d}x = \mathrm{e}^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d. \tag{2.1}$$

It follows that

$$p(t, x) = t^{-d/\alpha} p(1, t^{-1/\alpha}x), \qquad t > 0,\ x \in \mathbb{R}^d. \tag{2.2}$$

The above *scaling* implies in particular that

$$\int_{\{|x|<ct^{1/\alpha}\}} p(t, x)\mathrm{d}x = \int_{\{|x|<c\}} p(1, x)\mathrm{d}x > 0, \qquad c,\ t > 0. \tag{2.3}$$

It is well known that $p(1, x) \approx (1 + |x|)^{-d-\alpha}$ for $x \in \mathbb{R}^d$, see, e.g., [14, remarks after Theorem 21] or Kwaśnicki [48, (2.11)]. Here $\approx$ indicates that the ratio of both sides is bounded from above and below by a (strictly positive) constant. We call such comparisons *sharp*. Thus,

$$p(t,x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}, \qquad t>0,\ x\in\mathbb{R}^d. \tag{2.4}$$

By letting $p(t,x,y) := p(t,y-x)$, $x,y\in\mathbb{R}^d$, $t>0$, we introduce a (translation-invariant) transition density. On the space $D([0,\infty))$ of càdlàg functions (paths) $\omega : [0,\infty)\to\mathbb{R}^d$, we define the canonical process $Y_t(\omega)=\omega_t$, $t\geq 0$, and define Markovian measures $\mathbb{P}^x$, $x\in\mathbb{R}^d$, as follows. For (starting points) $x\in\mathbb{R}^d$, (times) $0<t_1<t_2<\ldots<t_n$ and (windows) $A_1, A_2,\ldots,A_n\subset\mathbb{R}^d$ we let

$$\mathbb{P}^x(\omega_{t_1}\in A_1,\ldots,\omega_{t_n}\in A_n) = \int\limits_{A_1}\mathrm{d}x_1\int\limits_{A_2}\mathrm{d}x_2\ldots\int\limits_{A_n}\mathrm{d}x_n\, p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_n-t_{n-1},x_{n-1},x_n).$$

By the Kolmogorov extension theorem, these finite-dimensional distributions uniquely determine $\mathbb{P}^x$, the law of the Markov process (Y_t) starting from x. We let $\mathbb{E}^x$ denote the corresponding expectation. As it turns out, Y is the isotropic α-stable process, a specific *symmetric Lévy process* in $\mathbb{R}^d$ with the Lévy triplet $(0,\nu,0)$, see, e.g., Sato [57, Section 11]. To analyze Y we use the standard complete right-continuous filtration $(\mathcal{F}_t, t\geq 0)$, see Protter [54, Theorem I.31]. In passing we also recall that every Lévy process is Feller, see [54] or Böttcher, Schilling and Wang [20], meaning that the operator semigroup

$$P(t)f(x)=\mathbb{E}^x f(Y_t),\quad x\in\mathbb{R}^d,\ t\geq 0, \tag{2.5}$$

leaves $C_0(\mathbb{R}^d)$, the space of continuous functions $f:\mathbb{R}^d\to R$ satisfying $f(x)\to 0$ as $|x|\to\infty$, invariant and is strongly continuous on that space. As in the Introduction, we write $\nu(x,y)=\nu(y-x)=c_{d,\alpha}|y-x|^{-d-\alpha}$ and $\nu(x,\mathrm{d}y)=\nu(x,y)\mathrm{d}y$ and for $u\colon\mathbb{R}^d\to\mathbb{R}$ and $x\in\mathbb{R}^d$ we define

$$\begin{aligned}\Delta^{\alpha/2}u(x) &= \lim_{\varepsilon\to 0^+}\int_{\{|y-x|>\varepsilon\}}\big[u(y)-u(x)\big]\nu(x,\mathrm{d}y) \\ &= \lim_{\varepsilon\to 0^+}\tfrac{1}{2}\int_{\{|z|>\varepsilon\}}\big[u(x+z)+u(x-z)-2u(x)\big]\nu(z)\,\mathrm{d}z.\end{aligned} \tag{2.6}$$

This is the fractional Laplacian (it is also common to use the notation $-(-\Delta)^{\alpha/2}$ for this operator). The limit exists, e.g., for $u\in C_c^\infty(\mathbb{R}^d)$, the smooth functions with compact support. The operator $\Delta^{\alpha/2}$ extends to the infinitesimal generator of the Feller semigroup defined by Eq. 2.5 on $C_0(\mathbb{R}^d)$. For a discussion of the many equivalent definitions of $\Delta^{\alpha/2}$ we refer to [48].

2.2 Exit Times

The *time of the first exit* of Y from an open set $U\subset\mathbb{R}^d$ is

$$\tau_U := \inf\{t>0:\ Y_t\notin U\}.$$

By translation invariance and scaling of $p(t,x,y)$, the law of $\{x+Y_t, t\geq 0\}$ under $\mathbb{P}^0$ is the same as the law of $\{Y_t, t\geq 0\}$ under $\mathbb{P}^x$ for $x\in\mathbb{R}^d$, and, under $\mathbb{P}^0$, the law of $\{cY_t\geq 0\}$ equals that of $\{Y_{c^\alpha t}\geq 0\}$. Hence, τ_{x+U} has the same law under $\mathbb{P}^0$ as τ_U under $\mathbb{P}^x$ and, under $\mathbb{P}^0$, $c\tau_U$ has the same law as $\tau_{c^\alpha U}$.

Recall that D is a nonempty bounded open subset of $\mathbb{R}^d$ which is Lipschitz (in the sense of [11, p. 156]). In particular, D is regular, meaning that

$$\mathbb{P}^x(\tau_D = 0) = 1 \quad \text{for} \quad x \in \partial D. \tag{2.7}$$

This follows from the radial symmetry of $p_t(x)$, Zaremba's exterior cone property of D and Blumenthal's 0-1 law, as in [25, Section 4.4]. We also refer to [5, VII.3, IV] for analytic definitions and treatment of regularity and to connections to the theory of stochastic processes. It is well known that the regularity of D is equivalent to solvability of the Dirichlet problem with arbitrary continuous data; see the above references. This condition will be important in Section 2.4.

The boundedness of D assures that the process leaves D in *finite time*: $\mathbb{P}^x(\tau_D < \infty) = 1$, see, e.g., Bogdan and Byczkowski [10, Subsection 2.3]. We consider Y_{τ_D}, the position at the exit time, and $Y_{\tau_D-} := \lim_{s\uparrow\tau_D} Y_s$, the position just before the exit. Thanks to the Lipschitz geometry of D,

$$\mathbb{P}^x(Y_{\tau_D} \in \partial D) = 0 \quad \text{for} \quad x \in D. \tag{2.8}$$

This is the principle of "not hitting the boundary upon the first exit" known since Bogdan [8, Lemma 6]; see also Bogdan, Grzywny, Pietruska-Pałuba and Rutkowski [12, Corollary A.2] for generalizations. In particular, the first exit from D occurs by a *jump*:

$$\mathbb{P}^x[\tau_D < \infty, Y_{\tau_D-} \neq Y_{\tau_D}] = 1, \qquad x \in D. \tag{2.9}$$

The random variable τ_D leads to important analytic objects, especially

$$p_D(t,x,y) := p(t,x,y) - \mathbb{E}^x[p(t-\tau_D, Y_{\tau_D}, y);\ \tau_D < t], \quad t > 0,\ x, y \in D,$$

the *Dirichlet heat kernel*, see, e.g., Chung and Zhao [26, Chapter 2.2]. The function is the transition density of the process Y *killed* upon exiting D. Namely,

$$\mathbb{E}^x[f(Y_t); t < \tau_D] = \int_D f(y) p_D(t,x,y) \mathrm{d}y, \quad x \in D,\ t > 0, \tag{2.10}$$

and the following Chapman–Kolmogorov equations hold for p_D:

$$\int_D p_D(s,x,z) p_D(t,z,y) \mathrm{d}z = p_D(t+s,x,y), \quad s,t > 0,\ x,y \in D. \tag{2.11}$$

It is also well known that $p_D(t,x,y)$ is jointly continuous and positive for all $(t,x,y) \in (0,\infty) \times D \times D$. In view of Eq. 2.9, the joint distribution of $(\tau_D, Y_{\tau_D-}, Y_{\tau_D})$ under $\mathbb{P}^x$ is given by the following *Ikeda–Watanabe formula*

$$\mathbb{P}^x[\tau_D \in I,\ Y_{\tau_D-} \in A,\ Y_{\tau_D} \in B] = \int_I \mathrm{d}s \int_A \mathrm{d}v \int_B \mathrm{d}z\, p_D(s,x,v)\nu(v,z), \tag{2.12}$$

where $x \in D$, $I \subset [0,\infty)$, $A \subset D$ and $B \subset D^c$, see, e.g., Bogdan, Rosiński, Serafin and Wojciechowski [18, Subsection 4.2]. Let

$$h_D(x,z) := \int_0^\infty \mathrm{d}s \int_D \mathrm{d}v\, p_D(s,x,v)\nu(v,z), \quad x \in D,\ z \in D^c. \tag{2.13}$$

By Eq. 2.12, the function is the density of the *harmonic measure* of D:

$$\mathbb{P}^x(Y_{\tau_D} \in B) = \int_B h_D(x,z)\mathrm{d}z, \qquad x \in D, \quad B \subset D^c. \tag{2.14}$$

By the same reason,

$$\int_0^\infty \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(s,x,v)\nu(v,z) = 1, \qquad x \in D. \tag{2.15}$$

The *survival probability* $\mathbb{P}^x(\tau_D > t)$ can be expressed in two ways as follows:

$$\mathbb{P}^x(\tau_D > t) = \int_t^\infty \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(s,x,v)\nu(v,z) \tag{2.16}$$

$$= \int_D p_D(t,x,y)\,\mathrm{d}y, \quad t > 0,\ x \in D. \tag{2.17}$$

Indeed, the first equation follows from the Ikeda–Watanabe formula Eq. 2.12, and the second from Eq. 2.10. Combining this with Eq. 2.15 yields, for all $t > 0$, $x \in D$,

$$\int_D p_D(t,x,y)\mathrm{d}y + \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(s,x,v)\nu(v,z) = 1. \tag{2.18}$$

We also have

$$\mathbb{P}^x(\tau_D \le t) = \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(s,x,v)\nu(v,z), \quad t > 0,\ x \in D. \tag{2.19}$$

For future use, we record the following fact on the survival probability.

Lemma 2.1 *If $F \subset D$ is compact and $T \in (0,\infty)$, then a constant $\eta = \eta(F,T) > 0$ exists, such that $\mathbb{P}^x(\tau_D > t) \ge \eta$ for all $x \in F$ and $0 < t \le T$.*

Proof Let $r = \mathrm{dist}(F, D^c)$. Of course, $0 < r < \infty$. We have $\mathbb{P}^x(\tau_D > t) \ge \mathbb{P}^x(\tau_{B(x,r)} > t) = \mathbb{P}^0(\tau_{B(0,r)} > t)$. The latter is clearly nonincreasing in t. It is also strictly positive, see, e.g., Chen and Song [22, Theorem 2.4]. □

For clarity we note that some of the arguments in [22, Theorem 2.4] refer to Chung and Zhao [26], who deal with the Brownian motion, but the arguments apply more generally. Alternatively, we may use the sharp explicit bounds for $\mathbb{P}^x(\tau_{B(0,r)} > t)$ given in Bogdan, Grzywny, Ryznar [13, Lemma 6].

2.3 The Killed Semigroup

In this section, we consider $(P_D(t), t > 0)$, the semigroup of the process killed upon leaving D. Thus,

$$P_D(t)f(x) := \mathbb{E}^x\big[f(Y_t)\mathbb{1}_{\{t<\tau_D\}}\big] = \int_D f(y)p_D(t,x,y)\mathrm{d}y, \quad x \in D. \tag{2.20}$$

It is a sub-Markovian semigroup on the space $B_b(D)$ of bounded measurable functions on D. By $C_0(D)$ we denote the space of continuous functions on D that vanish at ∂D. Recall that a semigroup $S(t)$ of kernel operators on $B_b(D)$ is called *Feller semigroup* if $S(t)C_0(D) \subset C_0(D)$, $t > 0$, and for every $f \in C_0(D)$ the *orbit* $t \mapsto S(t)f$ is continuous on $[0,\infty)$ in the supremum norm $\|\cdot\|_\infty$. It is called *strong Feller* if $S(t)B_b(D) \subset C_b(D)$, $t > 0$, where $C_b(D)$ refers to the space of bounded, continuous functions on D. $(S(t))$ is called C_b*-Feller* if $S(t)f \to f$ on compact subsets of D when $t \to 0$ and $f \in C_b(D)$, see Definition A.2 for details. Concerning the semigroup of the killed process, we have:

Lemma 2.2 *The semigroup* $(P_D(t))$ *is Feller, strong Feller, and* C_b*-Feller. Moreover,* $P_D(t)B_b(D) \subset C_0(D)$ *for every* $t > 0$.

Proof Due to [24, page 68], the semigroup P_D is a Feller semigroup and enjoys the strong Feller property, because these properties hold true for the unkilled semigroup P and D is regular. At this point [59, Lemma 3.1] implies that P_D is a C_b-semigroup. Note that [59] is actually concerned with Feller semigroups on $\mathbb{R}^d$, but the proof of [59, Lemma 3.1] applies to open subsets of $\mathbb{R}^d$, too. The last statement of Lemma 2.2 can be proved as in [26, Theorem 2.7]. □

For our development in Section 5, we need to characterize the C_b-generator of the C_b-Feller semigroup P_D; see Definition A.4 for the definition. The operator is a (typically strict) extension of the generator of the Feller semigroup on $C_0(D)$. Whereas the latter is defined as the derivative at $t = 0$ of the orbits with respect to the norm $\|\cdot\|_\infty$, the C_b-generator is defined via the Laplace transform of the semigroup. To this end we will use a general Theorem A.5, providing several equivalent characterizations of C_b-generators, and the following simple lemma.

Lemma 2.3 *There exists* $c = c(D, \alpha) > 0$ *such that if* $D \subset B(0, R)$*, then*

$$\mathbb{P}^x(\tau_D \le t; Y_{\tau_D} \in B(0, 2R)^c) \le cR^{-\alpha}\mathbb{P}^x(\tau_D \le t), \qquad x \in D,\ t > 0.$$

Proof We can find a constant c_1 such that $\nu(y, B(0, 2R)^c) \le c_1 R^{-\alpha}$ for $y \in B(0, R)$. By the Ikeda–Watanabe formula Eq. 2.12,

$$\begin{aligned}\mathbb{P}^x(\tau_D \le t, Y_{\tau_D} \in B(0, R)^c) &= \int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_D(s, x, y)\nu(y, B(0, R)^c)\\ &\le c_1 R^{-\alpha} \int_0^t \mathrm{d}s \int_D \mathrm{d}y\, p_D(s, x, y). \end{aligned} \tag{2.21}$$

On the other hand, by Eq. 2.19,

$$\mathbb{P}^x(\tau_D \le t) \ge c_2 \int_0^t ds \int_D dy\, p_D(s, x, y), \tag{2.22}$$

where $c_2 := \inf_{y \in D} \nu(y, D^c) > 0$. The result follows with $c = c_1/c_2$. □

The final ingredient to characterize the C_b-generator $\mathscr{A}^D$ of P_D in terms of the (unkilled) semigroup $P = P_{\mathbb{R}^d}$ is the *Dynkin operator* $\mathscr{D}$, defined as

$$\mathscr{D}u(x) := \lim_{x \to 0} \frac{\mathbb{E}^x u(Y_{\tau_{B(x,r)}}) - u(x)}{\mathbb{E}^x \tau_{B(x,r)}}, \tag{2.23}$$

for any function $u \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, for which the limit exists. We also denote by $\tilde{u}$ the extension of a function $u \in C_0(D)$ to $\mathbb{R}^d$ by zero.

Proposition 2.4 *Let* $u, f \in C_b(D)$*. Then the statement* $u \in D(\mathscr{A}^D)$ *and* $\mathscr{A}^D u = f$ *is equivalent to each of the following:*

(i) $\sup_{t \in (0,1)} \|t^{-1}(P_D(t)u - u)\|_\infty < \infty$ *and*

$$f(x) = \lim_{t \to 0} \frac{P_D(t)u(x) - u(x)}{t}, \qquad x \in D. \tag{2.24}$$

(ii) $u \in C_0(D)$ *and*

$$f(x) = \lim_{t \to 0} \frac{P(t)\tilde{u}(x) - \tilde{u}(x)}{t}, \qquad x \in D. \tag{2.25}$$

(iii) $u \in C_0(D)$ *and*

$$f(x) = \lim_{\varepsilon \to 0^+} \int_{\{|y-x|>\varepsilon\}} \big[\tilde{u}(y) - \tilde{u}(x)\big] \nu(x, y) \mathrm{d}y, \qquad x \in D. \tag{2.26}$$

Proof That $u \in D(\mathscr{A}^D)$ and $\mathscr{A}^D u = f$ is equivalent to (i) is an immediate consequence of the equivalence of (i) and (iii) in Theorem A.5. We shall prove (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i) $\Rightarrow$ (iii). In the proof of (i) $\Leftrightarrow$ (ii), we follow the ideas of [2, Theorem 2.3] which, however, is concerned with the space $C_0(D)$ instead of $C_b(D)$.

(i) $\Rightarrow$ (ii). By Lemma 2.2 we have $P_D(t) f \in C_0(D)$ for all $t > 0$. It follows that the Laplace transform of P_D takes values in $C_0(D)$ which, in turn, implies that $D(\mathscr{A}^D) \subset C_0(D)$. To prove Eq. 2.25, we compare the difference quotients in Eqs. 2.24 and 2.25. Arguing as in the proof of [2, Theorem 2.3], we see that

$$\begin{aligned}
&\big[P_D(t)u(x) - u(x)\big] - \big[P(t)\tilde{u}(x) - \tilde{u}(x)\big] = P_D(t)\tilde{u}(x) - P(t)\tilde{u}(x) \\
&= -\mathbb{E}^x\big[P(t-\tau_D)\tilde{u}(Y_{\tau_D}); \tau_D \le t\big] = \mathbb{E}^x\big[\tilde{u}(Y_{\tau_D}) - P(t-\tau_D)\tilde{u}(Y_{\tau_D}); \tau_D \le t\big] \\
&= \mathbb{E}^x\big[\tilde{u}(Y_{\tau_D}) - P(t-\tau_D)\tilde{u}(Y_{\tau_D}); \tau_D \le t,\ Y_{\tau_D} \in B(0, 2R)\big] \\
&\quad + \mathbb{E}^x\big[(\tilde{u}(Y_{\tau_D}) - P(t-\tau_D)\tilde{u}(Y_{\tau_D})); \tau_D \le t,\ Y_{\tau_D} \in B(0, 2R)^c\big] \\
&=: I_1 + I_2.
\end{aligned}$$

Given $\varepsilon > 0$, we may pick R so large that $D \subset B(0, R)$ and $R^{-\alpha} \le \varepsilon$. If $s \to 0$ then $P(s)\tilde{u} \to \tilde{u}$ uniformly on $\mathbb{R}^d$, in particular on $B(0, R)$. We may thus pick $t_0 > 0$ so that $|I_1| \le \varepsilon \mathbb{P}^x(\tau_D \le t)$ for all $t \le t_0$. As for I_2, we infer from Lemma 2.3 that

$$|I_2| \le 2\|u\|_\infty c R^{-\alpha} \mathbb{P}^x(\tau_D \le t).$$

By the choice of R, we see that

$$|I_1 + I_2| \le C\varepsilon \mathbb{P}^x(\tau_D \le t)$$

for $t \le t_0$. Pick $r > 0$ so that $B(x, r) \subset D$. Then $\mathbb{P}^x(\tau_D \le t) \le \mathbb{P}^x(\tau_{B(x,r)} \le t)$. By [20, Proposition 2.27(d) and Theorem 5.1] or Lévy inequality and [15, Remark 1], we have $\mathbb{P}^x(\tau_{B(x,r)} \le t) \le Mt$ for some $M \in (0, \infty)$ and all $t \le t_0$. Therefore for all such t,

$$\frac{|P_D(t)u(x) - P(t)\tilde{u}(x)|}{t} \le MC\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the difference quotient in Eq. 2.25 has the same limit as that in Eq. 2.24.

(ii) $\Rightarrow$ (i). Since $(\lambda - \mathscr{A}^D)^{-1}$ exists for $\lambda > 0$ and maps $D(\mathscr{A}^D)$ onto $C_b(D)$, this can be proved as in [2, Theorem 2.3].

(i) $\Rightarrow$ (iii). Adding an isolated point $\dagger$ as a cemetery state to D, we can consider the *stopped process* $(Y_{t\wedge\tau_D})_{t\ge 0}$ as a Markov process with state space $E = D \cup \{\dagger\}$. Here $Y_{t\wedge\tau_D} = \dagger$ for $t \ge \tau_D$. The transition semigroup of this process is P_D, if we extend a function g in $C_b(D)$ to E by setting $g(\dagger) = 0$. It follows from the implication (iii) $\Rightarrow$ (ii) in Theorem A.5, that for u, f as in (i) the pair (u, f) belongs to the *full generator* in the sense of [29, Equation (5.5) of Chapter 1]. By [29, Proposition 4.1.7], the process

$$u(Y_{t\wedge\tau_D}) - u(x) - \int_0^{t\wedge\tau_D} f(Y_s) \mathrm{d}s$$

is a martingale with respect to $\mathbb{P}^x$. Now consider the stopping time $\tau_{B(x,r)}$, where $r > 0$ is so small that $B(x,r) \subset D$. Then $\tau_{B(x,r)} \wedge \tau_D = \tau_{B(x,r)}$. Noting that u and f are bounded functions, it follows from optional stopping that

$$\mathbb{E}^x u(Y_{B(x,r)}) - u(x) = \mathbb{E}^x \int_0^{\tau_{B(x,r)}} f(Y_s)\mathrm{d}s.$$

Then, from the discussion of scaling in Subsection 2.1, we get

$$\begin{aligned}
&\frac{\mathbb{E}^x u(Y_{\tau_{B(x,r)}}) - u(x)}{\mathbb{E}^x \tau_{B(x,r)}} = \frac{1}{\mathbb{E}^x \tau_{B(x,r)}} \mathbb{E}^x \int_0^{\tau_{B(x,r)}} f(Y_s)\,\mathrm{d}s \\
&= \frac{1}{\mathbb{E}^x \tau_{B(x,1)}} \mathbb{E}^x r^{-\alpha} \int_0^{r^\alpha \tau_{B(x,1)}} f(Y_s)\,\mathrm{d}s = \frac{1}{\mathbb{E}^x \tau_{B(x,1)}} \mathbb{E}^x \int_0^{\tau_{B(x,1)}} f(Y_{r^\alpha t})\,\mathrm{d}t \\
&\to \frac{1}{\mathbb{E}^x \tau_{B(x,1)}} \mathbb{E}^x \tau_{B(x,1)} f(x)\,\mathrm{d}s = f(x) \quad \text{as } r \to 0,
\end{aligned}$$

because f is continuous and bounded and we can use the dominated convergence theorem. This shows that $\mathscr{D}u(x) = f(x)$, for all $x \in D$. At this point [48, Lemma 3.3] yields (iii).

(iii) ⇒ (ii). This follows from [48, Lemma 3.4]. □

2.4 The Dirichlet Problem

We are interested in the following operators

$$(H_D(\lambda)g)(x) := \int_0^\infty \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, e^{-\lambda s} p_s^D(x,v)\nu(v,z)g(z), \tag{2.27}$$

where $x \in D$, $\lambda \geq 0$, and g is a nonnegative or integrable function on D^c. From the Ikeda–Watanabe formula it is immediate that

$$H_D(\lambda)g(x) = \mathbb{E}^x\left[e^{-\lambda\tau_D} g(Y_{\tau_D})\right], \quad x \in D. \tag{2.28}$$

Lemma 2.5 *If $g \in B_b(D^c)$ then $H_D(\lambda)g \in C_b(D)$. If g is also continuous at ∂D, then $H_D(\lambda)g$ extends continuously to $\overline{D}$ and the extension equals g on ∂D.*

Proof The result is well known, but the following argument is of some interest. We have $H_D(0)\mathbb{1}_{D^c} = \mathbb{1}_D$ on D by Eq. 2.15. Therefore, as a consequence of the continuity (with respect to $x \in D$) of the integrand in Eq. 2.27 with $g \equiv 1$ and $\lambda = 0$, by Vitali's convergence theorem (see, e.g., [60, Chapter 22]), the integrand is uniformly integrable for x in every compact subset of D. By majorization, the integrand in Eq. 2.27 for general g and λ is also uniformly integrable, so $H_D(\lambda)g \in C_b(D)$.

Now assume that g is continuous at ∂D and let $x_0 \in \partial D$ and $x \in D$. Taking Eq. 2.15 into account again, we find

$$\begin{aligned}|H_D(\lambda)g(x) - g(x_0)| &\le \int_0^\infty \mathrm{d}t \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z \left(1 - e^{-\lambda t}\right) p_D(t,x,v)\nu(v,z)|g(x_0)| \\ &\quad + \int_0^\infty \mathrm{d}t \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, e^{-\lambda t} p_D(t,x,v)\nu(v,z)\big|g(z) - g(x_0)\big| \\ &\le \|g\|_\infty \int_0^\infty \mathrm{d}t \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z \left(1 - e^{-\lambda t}\right) p_D(t,x,v)\nu(v,z) \\ &\quad + \int_0^\infty \mathrm{d}t \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(t,x,v)\nu(v,z)\big|g(z) - g(x_0)\big| \\ &=: I_1(x) + I_2(x).\end{aligned}$$

For arbitrary $\delta > 0$ we get

$$I_1(x) = \|g\|_\infty\, \mathbb{E}^x\left[1 - e^{-\lambda\tau_D}\right] \le \|g\|_\infty\left[\mathbb{P}^x(\tau_D > \delta) + (1 - e^{-\lambda\delta})\right].$$

Recall that D is regular, whence $\limsup_{x\to x_0} \mathbb{P}^x(\tau_D > \delta) = 0$, see [24, (9)] or [26, Proposition 1.19]. So,

$$\limsup_{x\to x_0} I_1(x) \le \|g\|_\infty (1 - e^{-\lambda\delta}).$$

Since $\delta > 0$ was arbitrary, $I_1(x) \to 0$ as $x \to x_0$. We also have $I_2(x) = \mathbb{E}^x\big|g(X_{\tau_D}) - g(x_0)\big| \to 0$ as $x \to x_0$, since D is regular for the Dirichlet problem, see the discussion following Eq. 2.7. The proof is complete. ◻

Lemma 2.6 *If $\lambda \ge 0$, $g \in B_b(D^c)$ is continuous at ∂D and $h = H_D(\lambda)g$, then $h \in D(\mathscr{A}^D)$ and $\mathscr{A}^D h = \lambda h$.*

Proof As in the proof of Proposition 2.4, we make use of the Dynkin operator $\mathscr{D}$, defined by Eq. 2.23. Fix $x \in D$ and let $r > 0$ be so small that $B(x,r) \subset D$. To obtain a function defined on the whole of $\mathbb{R}^d$, we extend h by setting $h(x) = g(x)$ for $x \in D^c$. By Lemma 2.5, this yields a continuous function on $\overline{D}$. By the strong Markov property,

$$H_{B(x,r)}(\lambda)h(x) = H_{B(x,r)}(\lambda)H_D(\lambda)g(x) = H_D(\lambda)g(x) = h(x).$$

Thus,

$$\lim_{r\to 0} \frac{H_{B(x,r)(\lambda)}h(x) - h(x)}{\mathbb{E}^x\tau_{B(x,r)}} = 0. \tag{2.29}$$

On the other hand, from the discussion in Subsection 2.1,

$$\begin{aligned}&\frac{H_{B(x,r)}(0)h(x) - H_{B(x,r)}(\lambda)h(x)}{\mathbb{E}^x\tau_{B(x,r)}} = \frac{\mathbb{E}^x\left[(1 - e^{-\lambda\tau_{B(x,r)}})h(Y_{\tau_{B(x,r)}})\right]}{\mathbb{E}^x\tau_{B(x,r)}} \\ &= \frac{1}{\mathbb{E}^0\tau_{B(0,1)}}\mathbb{E}^0\left[r^{-\alpha}(1 - e^{-r^\alpha\lambda\tau_{B(0,1)}})h(x + rY_{\tau_{B(0,1)}})\right] \\ &\to \lambda h(x), \quad \text{as } r \to 0,\end{aligned} \tag{2.30}$$

because h is continuous by Lemma 2.5 and bounded, so we can use the dominated convergence theorem. Adding Eqs. 2.29 and 2.30, we get $\mathscr{D}h(x) = \lambda h(x)$ and the claim follows from [48, Lemma 3.3 and 3.4], including $h \in D(\mathscr{A}^D)$. ◻

3 The Transition Kernel with Reflections

In this section we define the transition kernel $k(t)$ aforementioned in the Introduction. To this end we define the kernel $\varphi : (0, \infty) \times D \times \mathscr{B}(D)$ by

$$\varphi(t, x, A) := \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(t, x, v)\nu(v, z)\mu(z, A). \tag{3.1}$$

Let us give an informal interpretation of φ. We consider $x \in D$ as the value of $Y_0 = X_0$, i.e., the starting point of both the processes; t as the value of τ_D, the first exit time of Y from D. By the Ikeda–Watanabe formula, $\int_D \mathrm{d}v\, p^D(t, x, v)\nu(v, z)$ is the density function of (τ_D, Y_{τ_D}). If the process X exists as described in the Introduction, then $\varphi(t, x, \mathrm{d}w)\mathrm{d}t$ is bound to be the joint distribution of (τ_D, X_{τ_D}). Of course, we mention X and Y only to develop intuition — in this paper we merely construct a specific transition density $k(t)$, using the analytic data: $p_D(t, x, y)$, $\nu(x, y)$ and $\mu(z, \cdot)$, but we do not analyze the process X, for which see [17].

We note some simple properties of the kernel φ.

Lemma 3.1 *Let φ be defined by Eq. 3.1. Then:*

(a) *For every $x \in D$, we have*

$$\int_0^\infty \mathrm{d}t\, \varphi(t, x, D) = 1.$$

(b) *For every $x \in D$ and $t > 0$, we have*

$$p_D(t, x, D) + \int_0^t \mathrm{d}s\, \varphi(s, x, D) = 1.$$

Proof Part (a) follows from Tonelli's theorem, the fact that every $\mu(z, \cdot)$ is a probability measure, and Eq. 2.15. The proof of (b) is similar, using Eq. 2.18 instead of Eq. 2.15. □

We shall use the operator S, defined as follows. For $f : (0, \infty) \times D \to [0, \infty]$,

$$Sf(t, x) := \int_0^t \mathrm{d}s \int_D \varphi(s, x, \mathrm{d}w) f(t - s, w), \quad t > 0, \quad x \in D.$$

By Lemma 3.1,

$$0 \le S\mathbb{1}(t, x) \le 1, \quad t > 0, \quad x \in D. \tag{3.2}$$

If $f : (0, \infty) \times D \times D \to [0, \infty]$, then we slightly abuse notation by also defining

$$Sf(t, x, y) := \int_0^t \mathrm{d}s \int_D \varphi(s, x, \mathrm{d}w) f(t - s, w, y), \quad t > 0, \quad x, y \in D.$$

We shall apply S to $f(t, x, y) = p_D(t, x, y)$. Given the interpretation of φ,

$$Sp_D(t, x, y) := \int_0^t \mathrm{d}s \int_D \varphi(s, x, \mathrm{d}w) p_D(t - s, w, y), \quad t > 0, \quad x, y \in D,$$

tentatively introduces a single reflection from D^c before time t. To accommodate more reflections, we iterate S and define

$$k(t, x, y) := \sum_{n=0}^\infty S^n p_D(t, x, y) \tag{3.3}$$

for $t > 0$, $x, y \in D$. Here, as usual, $S^0 := I$, the identity.

Given a set $A \subset D$ and a function $f : (0, \infty) \times D \times D \to [0, \infty)$ we let $f(t, x, A) = \int_A f(t, z, y)\mathrm{d}y$. Then Tonelli's theorem gives

$$Sf(t, x, A) = \int_0^t \mathrm{d}s \int_D \varphi(s, x, \mathrm{d}w) f(t-s, w, A), \quad t > 0, \quad x \in D.$$

Applying this to Eq. 3.3 and using Tonelli's theorem again, yields

$$k(t, x, A) := \sum_{n=0}^{\infty} S^n p_D(t, x, A). \tag{3.4}$$

Being a series of positive functions, k is well defined, with values in $[0, \infty]$. We also have the following Duhamel (or perturbation) formula:

$$k(t, x, A) = p_D(t, x, A) + Sk(t, x, A), \quad t > 0, \quad x \in D, \ A \in \mathscr{B}(D). \tag{3.5}$$

We shall gradually prove that k is a transition probability density. We first establish the Chapman–Kolmogorov equations.

Lemma 3.2 *For every $t, s > 0$, $x \in D$ and $A \in \mathscr{B}(D)$, we have*

$$\int_D k(t, x, \mathrm{d}y) k(s, y, A) = k(t+s, x, A).$$

Proof The equality may be obtained as in Bogdan, Hansen and Jakubowski [16, Lemma 2]. In fact, it is a special case of Bogdan and Sydor [19, Lemma 3], with the transition kernel $k(s, x, t, A)$ there equal to $p_D(t-s, x, A)$ and the perturbing kernel $J(u, z, \mathrm{d}u_1 \mathrm{d}z_1)$ given by $V(z, \mathrm{d}z_1)\mathbb{1}_{(u,\infty)}(u_1)\mathrm{d}u_1$, where $\mathrm{d}u_1$ is the Lebesgue measure on $\mathbb{R}$ and $V(x, A) := \int_{D^c} \nu(x, \mathrm{d}z)\mu(z, A)$. □

We next prove that k is sub-Markovian. Recall that by Lemma 3.1(b),

$$1 = p_D(t, x, D) + S\mathbb{1}(t, x), \quad t > 0, \ x \in D. \tag{3.6}$$

Lemma 3.3 *For all $t > 0$ and $x \in D$ we have $k(t, x, D) \leq 1$.*

Proof If $f : (0, \infty) \times D \to [0, 1]$, then by Lemma 3.1(b),

$$0 \leq p_D(t, x, D) + Sf(t, x) \leq p_D(t, x, D) + S\mathbb{1}(t, x) = 1.$$

Since $p_D(t, x, D) \leq 1$, it follows by induction that

$$\sum_{k=0}^{n+1} S^k p_D(t, x, D) = p_D(t, x, D) + S\Big(\sum_{k=0}^{n} S^k p_D\Big)(t, x, D) \leq 1.$$

By Eq. 3.4 and letting $n \to \infty$, we verify the claim. □

By iterating Eq. 3.6 we obtain the identity

$$1 = p_D(t, x, D) + Sp_D(t, x, D) + S^2\mathbb{1}(t, x), \quad t > 0, \ x \in D. \tag{3.7}$$

Making use of the lower bound in Hypothesis 1.1, we can actually establish that k is a Markovian kernel.

Theorem 3.4 *Under Hypothesis 1.1, $k(t, x, D) = 1$ for all $t > 0$, $x \in D$.*

Proof In view of Lemma 3.3, it suffices to prove that $k(t, x, D) \geq 1$ for all $t > 0$ and $x \in D$. We fix an arbitrary $T \in (0, \infty)$ and proceed in two steps.
Step 1: We prove a positive lower bound for $p_D(t, x, D) + Sp_D(t, x, D)$ uniform for $t \in (0, T)$ and $x \in D$. Let H and ϑ be as in Hypothesis 1.1(ii). By Lemma 2.1 and Eq. 2.17, we find $\eta > 0$ such that $p_D(s, w, D) \geq \eta$ for $s \in (0, T)$ and $w \in H$. Then for $t \in (0, T)$ and $x \in D$, by Eqs. 2.18 and 2.19 we get

$$\begin{aligned}
Sp_D(t,x,D) &= \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z \int_D p_D(s,x,v)\nu(v,z)\mu(z,\mathrm{d}w)p_D(t-s,w,D) \\
&\geq \eta \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(s,x,v)\nu(v,z)\mu(z,H) \\
&\geq \eta\vartheta \int_0^t \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(s,x,v)\nu(v,z) \\
&= \eta\vartheta \mathbb{P}^x(\tau_D < t) \\
&= \eta\vartheta \int_0^t \mathrm{d}s \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z \int_D p_D(s,x,y)\nu(y,z)\mu(z,\mathrm{d}w) \\
&\geq \eta\vartheta \int_0^t \mathrm{d}s \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z \int_D p_D(s,x,y)\nu(y,z)\mu(z,\mathrm{d}w)p_D(t-s,w,D) \\
&= \eta\vartheta Sp_D(t,x,D).
\end{aligned} \tag{3.8}$$

We conclude that all the above integrals are comparable. This will be quite useful later on, but for now we only deduce that for $t \in (0, T)$ and $x \in D$,

$$p_D(t,x,D) + Sp_D(t,x,D) \geq \mathbb{P}^x(\tau_D > t) + \eta\vartheta\mathbb{P}^x(\tau_D < t) \geq \eta\vartheta. \tag{3.9}$$

Step 2: We prove that $k(t, x, D) = 1$ for $t > 0$ and $x \in D$. Indeed, let

$$\ell = \inf\{k(t,x,D) : x \in D, t \leq T\}.$$

Clearly $0 \leq \ell \leq 1$. Iterating Eq. 3.5, for $t \in (0, T)$ and $x \in D$ we obtain

$$\begin{aligned}
k(t,x,D) &= p_D(t,x,D) + Sp_D(t,x,D) + S^2k(t,x,D) \\
&\geq p_D(t,x,D) + Sp_D(t,x,D) + \ell S^2\mathbb{1}(t,x) \\
&= \ell\big[p_D(t,x,D) + Sp_D(t,x,D) + S^2\mathbb{1}(t,x)\big] \\
&\quad + (1-\ell)\big[p_D(t,x,D) + Sp_D(t,x,D)\big].
\end{aligned}$$

By Eq. 3.7 and Step 1, $\ell \geq \ell + (1 - \ell)\eta\vartheta$, hence $\ell = 1$, which ends the proof. □

Corollary 3.5 We have $S^n\mathbb{1}(t, x) \leq (1 - \eta\vartheta)^{\lfloor n/2 \rfloor}$ for $n \in \mathbb{N}_0$, $t > 0$, $x \in D$.

Proof From Eqs. 3.7 and 3.9 we get $S^2\mathbb{1} \leq (1 - \eta\vartheta)\mathbb{1}$. Therefore, $S^{2n}\mathbb{1} \leq (1 - \eta\vartheta)^n\mathbb{1}$, $n \in \mathbb{N}$. The statement follows from this and Eq. 3.2. □

Theorem 3.4 and Eq. 3.4 yield

$$1 = p_D(t,x,D) + Sp_D(t,x,D) + S^2p_D(t,x,D) + \ldots, \quad t > 0,\ x \in D. \tag{3.10}$$

Corollary 3.5 shows that the series in Eq. 3.10 converges exponentially.

We will focus on the *semigroup* $K = (K(t), t > 0)$ associated to the transition kernels $(k(t), t > 0)$. More precisely, given $f \in B_b(D)$ we put

$$K(t)f(x) := \int_D f(y)k(t, x, \mathrm{d}y), \quad t > 0,\ x \in D. \tag{3.11}$$

As a consequence of Theorem 3.4, $K(t)$ is a Markovian operator on $B_b(D)$. It follows from Lemma 3.2 that the family $(K(t), t > 0)$ indeed satisfies the semigroup law $K(t+s) = K(t)K(s)$ for $s, t > 0$.

Remark 3.6 Many results of this section do not need the full strength of Hypothesis 1.1. Namely, Lemma 3.1, 3.2 and 3.3 do not require parts (ii) and (iii) of Hypothesis 1.1 and Theorem 3.4 and Corollary 3.5 do not use part (iii). The Lipschitz condition on D can be replaced throughout the paper by the regularity Eq. 2.7 and "not hitting the boundary" Eq. 2.8; this follows from our proofs. For instance, an open set $D \subset \mathbb{R}^d$ with the complement D^c satisfying the so-called volume density condition [12] is regular by the arguments following Eq. 2.7. If its boundary has zero Lebesgue measure, then Eq. 2.8 holds too; see [12, Corollary A.2]. The boundedness of D may be dropped in Lemma 3.1, 3.2, 3.3, Theorem 3.4, and Corollary 3.5, but it is crucial, e.g., in Section 6; see Remark 6.3.

Example 3.7 An important special case is when $\mu(z, A)$ does not depend on z. For instance, let $\mu(z, \cdot) = \delta_{x_0}$, the Dirac measure at a (fixed) point $x_0 \in D$. By Eq. 3.1, $\varphi(t, x, \mathrm{d}w) = (P_D(t)\kappa_D)(x)\delta_{x_0}(\mathrm{d}w)$, where $\kappa_D(v) := \nu(v, D^c)$ for $v \in D$. So, for $f \geq 0$, $t > 0$, and $x, y \in D$,

$$Sf(t, x, y) = \int_0^t \mathrm{d}s\, P_D(s)\kappa_D(x) f(t-s, x_0, y) = (P_D(\cdot)\kappa_D(x) * f(\cdot, x_0, y))(t),$$

where $*$ denotes the convolution on $\mathbb{R}$. In particular, for $t > 0$, $x, y \in D$,

$$Sp_D(t, x, y) = \int_0^t \mathrm{d}s \int_D \mathrm{d}v\, p_D(s, x, v)\kappa_D(v) p_D(t-s, x_0, y), \tag{3.12}$$

but it also follows that

$$k(\cdot, x, y) = p_D(\cdot, x_0, y) * \sum_{n=0}^{\infty} (P_D(\cdot)\kappa_D(x))^{*n}, \qquad x, y \in D. \tag{3.13}$$

Remark 3.8 The kernel k from Example 3.7 is quite singular. Indeed, we have $k(t, x, y) \geq Sp_D(t, x, y)$ and the inner (space) integral in Eq. 3.12 is

$$\int_D \mathrm{d}v\, p_D(s, x, v)\kappa_D(v) \geq c \int_D \mathrm{d}v\, p_D(s, x, v) = c\mathbb{P}^x(\tau_D > s) \geq c\mathbb{P}^x(\tau_D > s_0),$$

where $c := \inf\{\nu(v, D^c) : v \in D\} > 0$, $s \leq s_0 < \infty$, and we used Eq. 2.17. It follows that for $t \leq s_0$ and x in any given compact subset of D,

$$k(t, x, y) \geq C \int_0^t \mathrm{d}s\, p_D(t-s, x_0, y) = C \int_0^t \mathrm{d}s\, p_D(s, x_0, y),$$

where $C > 0$ by Lemma 2.1. Then, if y is close to x_0, $p_D(s, x_0, y) \approx p(s, x_0, y)$, so by [16, p. 249],

$$k(t, x, y) \geq C'|y - x_0|^{\alpha-d} \wedge \left(t^2 |y - x_0|^{-\alpha-d}\right).$$

Now, if $2\alpha < d$ and $f(y) \approx |y - x_0|^{-\alpha}$ on D, then we have $f \in L^2(D)$, but $\int_D k(t,x,y)f(y)\mathrm{d}y \equiv \infty$. So the semigroup $(K(t))$ does not even act on $L^2(D)$, in particular it is not associated to a Dirichlet form. Moreover, the kernels $k(t,x,y)$ are not symmetric, otherwise symmetry, the equality $K(t)\mathbb{1}_D = \mathbb{1}_D$, and Schur's test would make $K(t)$ a contraction on $L^2(D)$ for $t > 0$, a contradiction. We will also see in Remark 5.2 that $(K(t))$ is not a Feller semigroup. This motivates our approach by resolvent kernels in the next section.

4 The Laplace Transform of the Semigroup

We now study the Laplace transform $R(\lambda)$ of $(K(t))$, defined by

$$(R(\lambda)f)(x) := \int_0^\infty \mathrm{d}t \int_D e^{-\lambda t} k(t,x,\mathrm{d}y)f(y), \quad x \in D,\ \lambda > 0,$$

and relate it to the Laplace transform $R_D(\lambda)$ of $(P_D(t))$. To this end we introduce the operator $\Phi(\lambda)$,

$$(\Phi(\lambda)f)(x) := \int_0^\infty \mathrm{d}t \int_D e^{-\lambda t}\varphi(t,x,\mathrm{d}y)f(y), \quad x \in D,\ \lambda \geq 0,$$

where $f \in B_b(D)$. This operator is closely related to the operator $H_D(\lambda)$ from Section 2.4. Indeed, since μ is a kernel, for $f \in B_b(D)$ we may define

$$(\mu f)(z) := \mu(z,f) := \int_D \mu(z,\mathrm{d}y)f(y), \quad z \in D^c.$$

With this notation, we have $\Phi(\lambda)f = H_D(\lambda)[\mu f]$. From Lemma 2.5 we now obtain the following result about continuity of $\Phi(\lambda)f$. In the formulation of the result, we say that function $f \in C_b(D)$ *belongs to* $C(\overline{D})$ (and write $f \in C(\overline{D})$) if it has a (necessarily unique) continuous extension to $\overline{D}$. We then identify f and its extension to $\overline{D}$.

Lemma 4.1 *The operator $\Phi(\lambda)$ has the strong Feller property on D. If $f \in C_b(D)$ then $\Phi(\lambda)f \in C(\overline{D})$ and $\Phi(\lambda)f = \mu f$ on ∂D.*

Proof For $f \in B_b(D)$, $\mu f \in B_b(D^c)$ and Lemma 2.5 yields $\Phi(\lambda)f \in C_b(D)$, which is the strong Feller property. If $f \in C_b(D)$ then by Hypothesis 1.1(iii), the function μf is continuous at ∂D and Lemma 2.5 yields that $\Phi(\lambda)f \in C(\overline{D})$ and $\Phi(\lambda)f = \mu f$ at ∂D. □

We note that for $f \in B_b(D)$,

$$\Phi(\lambda)f(x) = \mathbb{E}^x\left[e^{-\lambda\tau_D}\mu f(Y_{\tau_D})\right], \quad x \in \overline{D}. \tag{4.1}$$

Lemma 4.2 *If $f \in B_b(D)$, $\lambda > 0$ and $\Phi(\lambda)f = f$, then $f = 0$.*

Proof By Lemma 4.1, $f = \Phi(\lambda)f \in C(\overline{D})$. Assume that $\sup_{\overline{D}} f > 0$. Note that $\sup_{D^c}\mu f \leq \sup_D f$. Using Eq. 4.1, for every $x \in D$ we get

$$\Phi(\lambda)f(x) \leq \mathbb{E}^x\left[e^{-\lambda\tau_D}\right]\sup_{\overline{D}} f < \sup_{\overline{D}} f,$$

because $\mathbb{P}^x$-$a.s.$ we have $\tau_D > 0$, by the right-continuity of the trajectories of the process Y. In particular, the maximum of f is attained at ∂D. Let H and ϑ be as in Hypothesis 1.1(ii).

Then $\sup_H f = (1-\delta)\sup_D f$ for some $\delta > 0$. As $\Phi(\lambda)f = f$, Lemma 4.1 yields for $z \in \partial D$,

$$
\begin{aligned}
f(z) = \mu f(z) &= \int_D f(x)\mu(z,\mathrm{d}x) \le (1-\delta)\mu(z,H)\sup_{\overline{D}} f + \mu(z, D\setminus H)\sup_{\overline{D}} f \\
&= \sup_{\overline{D}} f - \delta\mu(z,H)\sup_{\overline{D}} f \le (1-\delta\vartheta)\sup_{\overline{D}} f < \sup_{\overline{D}} f,
\end{aligned}
$$

a contradiction. So, $\sup_{\overline{D}} f \le 0$. By linearity, $\sup_{\overline{D}}(-f) \le 0$; $f = 0$ on D. □

Lemma 4.3 *We consider $\Phi(\lambda)$ as an operator on $C(\overline{D})$. Then the series*

$$
\sum_{n=0}^{\infty} \Phi(\lambda)^n
$$

converges in operator norm for $\lambda > 0$.

Proof It follows from Lemma 4.1, that $\Phi(\lambda)^2$ defines a strong Feller operator on $\overline{D}$. As is well known, see [55, §1.3], its square, i.e., $\Phi(\lambda)^4$, is an ultra-Feller operator, i.e., it maps bounded subsets of $B_b(\overline{D})$ to equicontinuous subsets of $C(\overline{D})$. In particular, $\Phi(\lambda)^4$ is a compact operator. By a variant of the Fredholm alternative, see [43, Theorem 15.4], $I - \Phi(\lambda)$ is invertible if and only if it is injective. The latter was proved in Lemma 4.2. So, 1 belongs to the resolvent set of $\Phi(\lambda)$. By the Krein–Rutman Theorem, see [58, Proposition V.4.1], the spectral radius $r(\Phi(\lambda))$ belongs to the spectrum of $\Phi(\lambda)$. Since $\|\Phi(\lambda)\| \le 1$ and 1 belongs to the resolvent set of $\Phi(\lambda)$, we must have $r(\Phi(\lambda)) < 1$, which is equivalent to the claim. □

We can now relate the resolvents $R(\lambda)$ and $R_D(\lambda)$.

Lemma 4.4 *For $\lambda > 0$ and $f \in B_b(D)$ we have $R(\lambda)f = \sum_{n=0}^{\infty}\Phi(\lambda)^n R_D(\lambda)f$. In particular, the identity $R(\lambda) = R_D(\lambda) + \Phi(\lambda)R(\lambda)$ holds true.*

Proof To prove the lemma, we make use of the series representation Eq. 3.4 for the kernel k. Let us first see how $\Phi(\lambda)$ interacts with the operator S. To that end, let $h : (0,\infty)\times D \to [0,\infty)$ and $x \in D$. By Tonelli's theorem,

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}t\, e^{-\lambda t} Sh(t,x) \\
&= \int_0^\infty \mathrm{d}t\, e^{-\lambda t}\int_0^t \mathrm{d}s \int_D \varphi(s,x,\mathrm{d}w)h(t-s,w) \\
&= \int_0^\infty \mathrm{d}s \int_s^\infty \mathrm{d}t\, e^{-\lambda t}\int_D \varphi(s,x,\mathrm{d}w)h(t-s,w) \\
&= \int_0^\infty \mathrm{d}s \int_D e^{-\lambda s}\varphi(s,x,\mathrm{d}w)\int_0^\infty dr\, e^{-\lambda r}h(r,w) \\
&= \Big(\Phi(\lambda)\int_0^\infty e^{-\lambda r}h(r,\cdot)\,\mathrm{d}r\Big)(x).
\end{aligned}
$$

Summarizing, we obtain the Laplace transform (in λ) of Sh by applying $\Phi(\lambda)$ to the Laplace transform of h. By this observation and induction, Eq. 3.4 yields

$$
R(\lambda)f = \sum_{k=0}^{\infty}\Phi(\lambda)^k R_D(\lambda)f,
$$

as claimed. □

Example 4.5 For μ in Example 3.7, $(\Phi(\lambda)f)(x) = f(x_0)\,(R_D(\lambda)\kappa_D)\,(x)$, so

$$R(\lambda)f(x) = R_D(\lambda)f(x) + R_D(\lambda)\kappa_D(x)\,R_D(\lambda)f(x_0)\sum_{n=0}^{\infty} R_D(\lambda)\kappa_D(x_0)^n$$
$$= R_D(\lambda)f(x) + R_D(\lambda)\kappa_D(x)\,R_D(\lambda)f(x_0)/\left(1 - R_D(\lambda)\kappa_D(x_0)\right),$$

for $x \in D$, $\lambda > 0$, and $f \in B_b(D)$.

We now come to the main result of this section, in which we characterize the closure of the range of $R(\lambda)$. Given a function $f \in C_b(D)$, we let

$$f_\mu(x) := \begin{cases} f(x), & \text{for } x \in D, \\ \mu(x, f), & \text{for } x \in D^c, \end{cases} \tag{4.2}$$

and we define the space $C_\mu(D)$ by

$$C_\mu(D) := \{f \in C_b(D) : f_\mu \text{ is continuous on } \overline{D}\}. \tag{4.3}$$

By Hypothesis 1.1(iii), the map $D^c \ni x \mapsto \mu(x, f)$ is continuous on ∂D. Thus, the condition that f_μ is continuous on $\overline{D}$ is equivalent with $f(x_n) \to \mu(x, f)$ whenever $(x_n) \subset D$ converges to $x \in \partial D$. We can therefore, similar to the remarks preceding Lemma 4.1, identify $C_\mu(D)$ with the following (closed) subspace of $C_b(\overline{D})$:

$$C_\mu(\overline{D}) := \left\{f \in C_b(\overline{D}) : f(z) = \int_D f(x)\mu(z, \mathrm{d}x) \text{ for all } z \in \partial D\right\}. \tag{4.4}$$

Example 4.6 Let $D = B(0, 1) \subset \mathbb{R}^d$ be the ball of radius 1 centered at 0 and $\mu(z, \cdot) = \delta_0$ for $z \in D^c$; see Example 3.7. In this case,

$$C_\mu(\overline{D}) = \{f \in C_b(\overline{D}) : f(z) = f(0) \text{ for all } z \in \partial D\}$$

and the extension f_μ is given by $f_\mu(x) = f(0)$ for $x \in D^c$.

Example 4.7 Let $D = B(0, 1)$ be as in Example 4.6, but let

$$\mu(z, \cdot) = \Big(\frac{1}{2} + \frac{1}{2|z|}\Big)\delta_0 + \Big(\frac{1}{2} - \frac{1}{2|z|}\Big)\delta_{(1-|z|^{-1})\frac{z}{|z|}}, \qquad z \in D^c.$$

Then μ satisfies Hypothesis 1.1. Note that $\mu(z, \cdot) = \delta_0$ when $|z| = 1$, so the space $C_\mu(\overline{D})$ is the same as in Example 4.6, but the extension f_μ is different:

$$f_\mu(x) = \Big(\frac{1}{2} + \frac{1}{2|x|}\Big)f(0) + \Big(\frac{1}{2} - \frac{1}{2|x|}\Big)f\Big((1 - |x|^{-1})\frac{x}{|x|}\Big), \quad x \in D^c.$$

Example 4.8 Let $\mu(z, \cdot) = \lambda_d(D)^{-1}\lambda_d$, where λ_d is the d-dimensional Lebesgue measure on D. Denote $\bar{f} := \lambda(D)^{-1}\int_D f(x)\mathrm{d}x$. Then $f_\mu(x) = \bar{f}$, $x \in D^c$, and

$$C_\mu(\overline{D}) = \{f \in C_b(\overline{D}) : f(z) = \bar{f} \text{ for all } z \in \partial D\}.$$

Note that we may rephrase the second statement of Lemma 4.1 by saying that $\Phi(\lambda)f \in C_\mu(D)$ for all $f \in C_b(D)$.

Theorem 4.9 *For $\lambda > 0$, the closure of the range of $R(\lambda)$ equals $C_\mu(D)$.*

Proof Let us first prove that the range of $R(\lambda)$ is contained in $C_\mu(D)$. To that end, fix $f \in B_b(D)$. As $P_D(t)f \in C_0(D)$ we have $R_D(\lambda)f \in C_0(D) \subset C(\overline{D})$ for any $f \in B_b(D)$. Using Lemma 4.1 and induction, we have $\Phi(\lambda)^k R_D(\lambda)f \in C_\mu(D) \subset C(\overline{D})$ for all $k \geq 1$ and Lemma 4.3 and Lemma 4.4 imply that $R(\lambda)f \in C(\overline{D})$.

Now fix $x_0 \in \partial D$. Putting $u = R(\lambda)f$, we find

$$\begin{aligned} u(x_0) = R(\lambda)f(x_0) &= R_D(\lambda)f(x_0) + \sum_{k=1}^{\infty}\Big(\Phi(\lambda)^k R_D(\lambda)f\Big)(x_0) \\ &= 0 + \sum_{k=1}^{\infty}\mu\big(x_0, \Phi(\lambda)^{k-1}R_D(\lambda)f\big) \\ &= \mu\Big(x_0, \sum_{k=1}^{\infty}\Phi(\lambda)^{k-1}R_D(\lambda)f\Big) = \mu(x_0, u). \end{aligned}$$

Here the second equality uses Lemma 4.4, the third Lemma 4.1 and the fact that $R_D(\lambda)f \in C_0(D)$. The fourth equality uses dominated convergence and the last Lemma 4.4 again. This shows that $u \in C_\mu(D)$. As the latter is closed and contains the range of $R(\lambda)$, it also contains the closure of the range.

To prove the converse, we only need to show that the range of $R(\lambda)$ is dense in $C_\mu(D)$. To that end, let $f \in C_\mu(D)$ and $g := f - \Phi(\lambda)f$. By Lemma 4.1, $g \in C_0(D)$. Since the semigroup of the killed process is strongly continuous on $C_0(D)$, the domain of its generator is dense in $C_0(D)$. We thus find a bounded sequence $(u_n) \subset C_0(D)$ such that $R_D(\lambda)u_n \to g$ with respect to $\|\cdot\|_\infty$.

Next observe that

$$\begin{aligned} \sum_{k=0}^{N}\Phi(\lambda)^k R_D(\lambda)u_n \to \sum_{k=0}^{N}\Phi(\lambda)^k g& \\ &= \sum_{k=0}^{N}\big(\Phi(\lambda)^k f - \Phi(\lambda)^{k+1}f\big) = f - \Phi(\lambda)^{N+1}f \end{aligned}$$

as $n \to \infty$. Given $\varepsilon > 0$, we may, as a consequence of Lemma 4.3, pick N so large, that $\sum_{k\geq N}\|\Phi(\lambda)^k\| \leq \varepsilon$. Taking Lemma 4.4 into account, we get

$$\begin{aligned} \limsup_{n\to\infty}\|R(\lambda)u_n - f\|_\infty &\leq C\varepsilon\|R_D(\lambda)\|_\infty + \|\Phi(\lambda)^{N+1}f\|_\infty \\ &\leq C\|R_D(\lambda)\|_\infty\varepsilon + \varepsilon\|f\|_\infty. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we see that $R(\lambda)u_n \to f$. □

5 The Generator of the Semigroup

Proposition 5.1 *$K(t)f \to f$ uniformly as $t \to 0$ if, and only if, $f \in C_\mu(D)$.*

Proof As is well known (see, e.g. [45, Remark 2.5]), $K(t)f \to f$ in the norm $\|\cdot\|_\infty$ if f belongs to the domain of the generator of K, i.e., the range of $R(\lambda)$. By Theorem 4.9, the latter is a dense subset of $C_\mu(D)$ so a standard approximation argument shows that the same is true for every $f \in C_\mu(D)$.

To see the converse, let $X := \{f \in B_b(D) : K(t)f \to f \text{ as } t \to 0\}$. Then X is a closed subspace of $B_b(D)$ that is invariant under the semigroup K. Moreover, the restriction of $(K(t))$ to X is strongly continuous. By the Hille–Yosida theorem, the generator of the restriction to X, which is nothing more than the part of the full generator in X, is dense in X. But then X must be contained in the closure of the range of $R(\lambda)$, i.e., $C_\mu(D)$. □

Remark 5.2 It follows from Proposition 5.1 that the semigroup K is *not*, in general, a Feller semigroup, since typically $C_0(D) \not\subset C_\mu(D)$, whence the orbits of functions in $C_0(D)$ are not $\|\cdot\|_\infty$-continuous.

We can now prove the first main result of this section.

Theorem 5.3 *K is a C_b-semigroup and has the strong Feller property.*

Proof Let $f \in B_b(D)$ and $x \in D$. From Eq. 3.5, we obtain

$$(K(t)f)(x) = (P_D(t)f)(x) + \int_D Sk(t,x,y)f(y)\mathrm{d}y. \tag{5.1}$$

Let us consider the second term on the right hand side of Eq. 5.1. We have

$$\begin{aligned}\Big|\int_D Sk(t,x,y)f(y)\,dy\Big| &\le \int_0^t \mathrm{d}s \int_D \mathrm{d}w\,\varphi(s,x,w)k(t-s,w,D)\|f\|_\infty \\ &= \|f\|_\infty \int_0^t \mathrm{d}s \int_D \mathrm{d}w\,\varphi(s,x,w) = \|f\|_\infty \mathbb{P}^x(\tau_D \le t), \end{aligned} \tag{5.2}$$

where the last equality uses Lemma 3.1(b) and Equation Eq. 2.17. By [24, Lemma 2], the latter converges to 0 as $t \to 0$, uniformly on compact subsets of D.

Let now $f \in C_b(D)$. We have seen that the integral in Eq. 5.1 converges locally uniformly to 0 as $t \to 0$. By Lemma 2.2, $P_D(t)f$ converges locally uniformly to f. Thus, Eq. 5.1 yields that $K(t)f \to f$ locally uniformly as $t \to 0$.

Let us now prove the strong Feller property. To that end, fix $t > 0$, $x \in D$ and $f \in B_b(D)$. Note that for $s \in (0,t)$ we have $K(t)f = K(s)K(t-s)f$. We set $g_s := K(t-s)f$. By Eq. 5.1 (with f replaced by g_s and t replaced by s) and Eq. 5.2,

$$|K(t)f(x) - P_D(s)g_s(x)| \le \|g_s\|_\infty \mathbb{P}^x(\tau_D \le s) \le \|f\|_\infty \mathbb{P}^x(\tau_D \le s).$$

The latter converges locally uniformly to 0 as $s \to 0$, so $P_D(s)g_s$ converges locally uniformly to $K(t)f$. Since P_D has the strong Feller property (Lemma 2.2), the functions $P_D(s)g_s$ are continuous. But then so is $K(t)f$. □

We can now characterize the C_b-generator $\mathscr{A}$ of the semigroup K. In the following theorem, we use the notation u_μ introduced in Eq. 4.2.

Theorem 5.4 *For $u, f \in C_b(D)$, the following are equivalent:*

(i) *$u \in D(\mathscr{A})$ and $\mathscr{A}u = f$.*
(ii) *$u \in C_\mu(D)$ and*

$$f(x) = \lim_{t\to 0} \frac{P(t)u_\mu(x) - u_\mu(x)}{t}, \qquad x \in D.$$

(iii) *$u \in C_\mu(D)$ and*

$$f(x) = \lim_{\varepsilon\to 0^+} \int_{\{|y-x|>\varepsilon\}} \big[u_\mu(y) - u_\mu(x)\big]\nu(x,y)\mathrm{d}y, \qquad x \in D.$$

(iv) *$u \in C_\mu(D)$ and, with γ given by Eq. 1.1,*

$$f(x) = \lim_{\varepsilon \to 0^+} \int_{\{|y-x|>\varepsilon\} \cap D} (u(y) - u(x))\gamma(x, \mathrm{d}y), \qquad x \in D.$$

Proof By Lemma 4.4, (i) is equivalent with $u = [R(\lambda)](\lambda u - f) = R_D(\lambda)(\lambda u - f) + \Phi(\lambda)u$. Thus (i) is equivalent to $u - \Phi(\lambda)u = R_D(\lambda)(\lambda u - f)$, which, in turn, is equivalent to $u - \Phi(\lambda)u \in D(\mathscr{A}^D)$ and $\mathscr{A}^D(u - \Phi(\lambda)u) = f - \lambda\Phi(\lambda)u$. By Theorem 4.9, $D(\mathscr{A}) \subset C_\mu(D)$, so the equivalence of (i) and (ii) follows from Proposition 2.4 and Lemma 2.6, applied with $g = \mu f$.

To prove the implication (i) $\Rightarrow$ (iii), we note that, by Proposition 2.4 and (the proof of) Lemma 2.6, the function $R_D(\lambda)(\lambda u - f) + \Phi(\lambda)u$ belongs to the domain of the Dynkin operator and $\mathscr{D}\big[R_D(\lambda)(\lambda u - f) + \Phi(\lambda)u\big] = f - \lambda\Phi(\lambda)u + \lambda\Phi(\lambda)u = f$ on all of D. At this point, [48, Lemma 3.3] yields (iii).

The implication (iii) $\Rightarrow$ (ii) follows once again from [48, Lemma 3.4], whereas (iv) is merely a reformulation of (iii). □

Remark 5.5 The condition $u \in C_\mu(D)$ that appears in the statements (ii) – (iv) of Theorem 5.4 can be seen as a *boundary condition* that a function $u \in C_b(D)$ necessarily satisfies if it belongs to $D(\mathscr{A})$. The condition is equivalent to

$$\lim_{D \ni x \to z} u(x) = \int_D u(y)\mu(z, \mathrm{d}y), \qquad z \in \partial D.$$

Corollary 5.6 Let γ be given by Eq. 1.1 and fix $\lambda > 0$. Then for every $f \in C_b(D)$ there exists a unique function $u \in C_\mu(D)$ satisfying

$$\lambda u(x) - \lim_{\varepsilon \to 0^+} \int_{\{|y-x|>\varepsilon\} \cap D} (u(y) - u(x))\gamma(x, \mathrm{d}y) = f(x), \quad x \in D.$$

Proof Since $\mathscr{A}$ is the generator of a C_b-Feller semigroup, $(0, \infty)$ belongs to the resolvent set of $\mathscr{A}$. Thus, for every $\lambda > 0$ and $f \in C_b(D)$ the equation $\lambda u - \mathscr{A}u = f$ has a unique solution $u := R(\lambda) \in D(\mathscr{A})$. Now the claim follows from the characterization of $\mathscr{A}$ in Theorem 5.4(iv). □

6 Asymptotic Behavior of the Semigroup

In order to establish the existence of an invariant measure, we employ the lower-bound technique of Lasota [49, Theorem 6.1]. Here is the first step.

Lemma 6.1 *Fix $t > 0$ and let H be as in Hypothesis 1.1(ii).*

(a) There is $\delta > 0$ such that $k(t, x, y) \geq \delta$ for all $x \in D$, $y \in H$.
(b) There is $\delta > 0$ such that $k(s, x, H) \geq \delta|H|$ for all $x \in D$ and $s \geq t$.
(c) There is $\varepsilon > 0$ such that $\int_D |k(t, x_1, y) - k(t, x_2, y)|\mathrm{d}y \leq 2 - \varepsilon$ for all $x_1, x_2 \in D$.

Proof (a) Since p_D is continuous and positive, by compactness there is a constant $c = c(D, H, t, \alpha) > 0$ such that $p_D(r, w, y) \geq c$ for all $r \in [t/2, t]$ and $w, y \in H$. Then, for

$x \in D$, $y \in H$, we get

$$\begin{aligned} k(t,x,y) &\geq Sp_D(t,x,y) \\ &\geq \int_0^{t/2} \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z \int_H p_D(s,x,v)\nu(v,z)\mu(z,\mathrm{d}w)p_D(t-s,w,y) \\ &\geq c\vartheta \int_0^{t/2} \mathrm{d}s \int_D \mathrm{d}v \int_{D^c} \mathrm{d}z\, p_D(s,x,v)\nu(v,z) = c\vartheta \mathbb{P}^x(\tau_D < t/2). \end{aligned} \tag{6.1}$$

Note that $x \mapsto p_D(s,x,v)\nu(v,z)$ is strictly positive and continuous for almost all triplets (s,v,z). Fatou's lemma implies that the function $x \mapsto \mathbb{P}^x(\tau_D < t/2)$ is lower semicontinous. Because of the regularity Eq. 2.7, the function tends to 1 as x approaches the boundary (see [26, Theorem 1.23]), so at some point of D it attains its minimum, say $C > 0$. Thus $k(t,x,y) \geq cC\vartheta =: \delta$ for all $x \in D$, $y \in H$.

(b) By the Chapman–Kolmogorov equations in Lemma 3.2, for $s > t$ we get

$$k(s,x,H) = \int_D k(s-t,x,y)k(t,x,H)\mathrm{d}y \geq \delta|H|.$$

(c) Pick again δ as in (a). By making H larger, we may assume that $|H| > 0$. Then, for all $x_1, x_2 \in D$ we have

$$\begin{aligned} &\int_D |k(t,x_1,y) - k(t,x_2,y)|\mathrm{d}y \\ &= \int_{D\setminus H} |k(t,x_1,y) - k(t,x_2,y)|\mathrm{d}y + \int_H |k(t,x_1,y) - \delta - (k(t,x_2,y) - \delta)|\mathrm{d}y \\ &\leq \int_{D\setminus H} k(t,x_1,y)\mathrm{d}y + \int_{D\setminus H} k(t,x_2,y)\mathrm{d}y \\ &\quad + \int_H (k(t,x_1,y) - \delta)\mathrm{d}y + \int_H (k(t,x_1,y) - \delta)\mathrm{d}y = 2 - 2\delta|H|, \end{aligned}$$

so we can take $\varepsilon = 2\delta|H| > 0$. □

We next consider the adjoint of the operators $K(t)$, $t > 0$. For a finite measure κ on D we put

$$K(t)^*\kappa(A) := \int_D k(t,x,A)\kappa(\mathrm{d}x), \qquad t > 0, \quad A \subset D.$$

Then $K(t)^*\kappa$ is again a measure on D and, by Tonelli's theorem and Theorem 3.4, $K(t)^*\kappa(D) = \kappa(D)$. Moreover, $K(t)^*\kappa$ is absolutely continuous with respect to Lebesgue measure, so we may think of $K(t)^*$ as operating on $L^1(D)$. With this in mind, the operators $K(t)^*$ are Markov operators in the sense of Lasota and Mackey [50], Lasota and York [51] and Komorowski [42]. We call a probability measure κ a *stationary distribution* if $K(t)^*\kappa = \kappa$ for all $t > 0$. Then, again, $K(t)^*\kappa$ is absolutely continuous with respect to the Lebesgue measure so that any stationary distribution must have a density, called *stationary density*.

Theorem 6.2 *There is a unique stationary distribution* κ. *Moreover, there exist* $M, \omega \in (0,\infty)$ *such that for every probability measure* ν *on* D,

$$\|K(t)^*\nu - \kappa\|_{\mathsf{TV}} \leq Me^{-\omega t}, \quad t > 0.$$

Proof By Lemma 6.1(b), $\liminf_{T\to\infty} \frac{1}{T}\int_0^T K(t,x,H)\mathrm{d}t > 0$ for $x \in D$. This is the lower bound mentioned at the beginning of the section. By Da Prato and Zabczyk [27, Remark 3.1.3], we get a stationary distribution. Since the proof in not given in [27, Remark 3.1.3] we refer the reader to Lasota and York [51, Theorem 3.1], Lasota [49, Theorem 6.1] or Komorowski [42, Theorem 3.1] for the proofs in the discrete-time case.

In view of Lemma 6.1(c), [44, Theorem 1.3] (see also [35]), applied to $P = K(1)$ yields the existence of constants $c > 0$ and $e^{-\omega} = \gamma > 0$ such that $\|K(n)^*\nu - \kappa\|_{\mathsf{TV}} \le c\gamma^n$ for all $n \in \mathbb{N}$ and probability measures ν on D. An arbitrary number $t > 0$ can be written as $t = n + r$, where $n \in \mathbb{N}_0$ and $r \in [0,1)$. Then,

$$\begin{aligned}\|K(t)^*\nu - \kappa\|_{\mathsf{TV}} &= \|K(r)^*K(n)^*\nu - K(r)^*\kappa\|_{\mathsf{TV}} \le \|K(n)^*\nu - K(r)^*\kappa\|_{\mathsf{TV}}\\ &\le ce^{-\omega n} \le ce^{\omega}e^{-\omega t} =: Me^{\omega t}.\end{aligned}$$

□

Remark 6.3 Here are some observations and open problems.

(i) The compactness of $\overline{D}$ is crucial in this section; Lemma 6.1 and Theorem 6.2 easily fail for $D = \mathbb{R}^d$, since then $K(t) = P(t)$. On the other hand, we conjecture that they hold if the boundedness of D is replaced by that of $x \mapsto \mathbb{E}_x\tau_D$, e.g., if D is an (unbounded) right circular cylinder.

(ii) The repulsion kernel μ enters the construction of $k(t)$ only through the *return kernel* $V(x, A) := \int_{D^c} \nu(x,z)\mu(z,A)\mathrm{d}z$. We expect results for kernels V on D satisfying $V(\cdot, D) \le \nu(\cdot, D^c)$ describing general reflections.

(iii) It is interesting to apply our methods to study semigroups of resurrections or recurrent extensions of Markov processes with scaling, see [32, 37, 40, 53, 56]. In the same vein, it is important to study the semigroup corresponding to the quadratic form and the Neumann problem in [28]; see also [12, 65] for motivation. In these settings, Hypothesis 1.1(ii) fails.

Appendix

A. C_b-Feller semigroups

By Remark 5.2, the semigroup $(K(t), t > 0)$ is, in general, *not* a Feller semigroup, so in this paper we use a different semigroup concept, namely the notion of C_b-Feller semigroup. This can be seen as a special case of the theory of "semigroups on norming dual pairs", introduced in [45, 46]. As this is not a standard notion, we introduce this concept in this appendix and reformulate the relevant results from [45, 46] in our special case.

Throughout, we let $E \subset$ be an open subset of $\mathbb{R}^d$, or, more generally, a Polish space. A *kernel* on E is a map $k : E \times \mathscr{B}(E) \to \mathbb{C}$ such that (i) the map $x \mapsto k(x, A)$ is measurable for all $A \in \mathscr{B}(E)$ (ii) the map $A \mapsto k(x, A)$ defines a measure on E for every $x \in E$ and (iii) we have $\sup_x |k|(x, E) < \infty$, where $|k|(x, \cdot)$ refers to the total variation of $k(x, \cdot)$.

A bounded linear operator T on $C_b(E)$ is called a *kernel operator*, if there exists a kernel k such that

$$(Tf)(x) = \int_E f(y)\,k(x, \mathrm{d}y) \qquad f \in C_b(E),\ x \in E. \tag{A.1}$$

As it turns out, being a kernel operator can be characterized by an additional continuity condition with respect to the weak topology $\sigma := \sigma(C_b(E), \mathscr{M}(E))$ induced by the space of bounded (complex/signed) measures. We note that for a sequence of functions $(f_n) \subset C_b(E)$, the convergence with respect to σ is nothing else than *bp-convergence* (bp is short for bounded, pointwise), which means $\sup_n \|f_n\|_\infty < \infty$ and $f_n \to f$ pointwise. Indeed, that bp-convergence implies σ-convergence follows from the dominated convergence theorem whereas the converse implication follows easily using the uniform boundedness principle.

Lemma A.1 *Let $T \in \mathscr{L}(C_b(E))$ be a bounded linear operator. The following are equivalent:*

(i) T is a kernel operator;
(ii) T is σ-continuous;
(iii) If $(f_n) \subset C_b(E)$ bp-converges to $f \in C_b(E)$, then Tf_n bp-converges to Tf.

Proof The equivalence of (i) and (ii) is proved in [46, Proposition 3.5] and the implication (ii) $\Rightarrow$ (iii) is trivial in view of the above comment. To see (iii) $\Rightarrow$ (i), let $\varphi(f) = (Tf)(x)$. By (iii), it follows that $\varphi(f_n) \to 0$ whenever f_n bp-converges to 0. Now [7, Theorem 7.10.1] implies that $\varphi(f) = \int_E f\, d\nu_x$ for some Baire (hence Borel, as E is Polish) measure ν_x. The measurable dependence of ν_x on x can now be proved in a standard way, see the proof the implication (i) $\Rightarrow$ (ii) in [46, Proposition 3.5]. □

In what follows, the space of bounded, σ-continuous operators (equivalently: kernel operators) on $C_b(E)$ is denoted by $\mathscr{L}(C_b(E), \sigma)$. Note that any operator $T \in \mathscr{L}(C_b(E), \sigma)$ can uniquely be extended to a bounded linear operator on all of $B_b(E)$, by merely plugging $f \in B_b(E)$ into the right hand side of Eq. A.1. In what follows we do not distinguish between T and its extension to $B_b(E)$.

We are now ready to define what a C_b-semigroup is. To simplify the exposition, we restrict ourselves to sub-Markovian semigroups, as all the semigroups appearing in this article have this property. Obviously, a kernel operator T with kernel k is (sub-)Markovian if and only if the kernel k is (sub-)Markovian, i.e. $k(x, \cdot)$ is a (sub-)probability measure for every $x \in E$.

Definition A.2 A *C_b-Feller semigroup* is a family $(T(t), t > 0) \subset \mathscr{L}(C_b(E), \sigma)$ with the following properties:

(i) $T(t)$ is a sub-Markovian kernel operator for every $t > 0$;
(ii) $T(t+s) = T(t)T(s)$ for all $t, s > 0$;
(iii) for $f \in C_b(E)$ we have $T(t)f \to f$ as $t \to 0$, uniformly on compact subsets of E.

In case that E is locally compact, it follows along the lines of [59, Lemma 3.1], which is concerned with the case $E = \mathbb{R}^d$, that a Feller semigroup on $C_0(E)$ can be extended to a C_b-Feller semigroup on $C_b(E)$. We should point out, however, that a C_b-Feller semigroup in the above sense does not necessarily leave the space $C_0(E)$ invariant. In that respect, our definition of C_b-Feller semigroup slightly differs from that in [38, Definition 4.8.6] where a C_b-Feller semigroup is assumed to be Feller.

Recalling the connection between bp-convergence and σ-convergence, we see that the requirement (iii) in the above definition in particular implies that $T_t f \to f$ as $t \to 0$ with respect to σ and thus, by the semigroup law and the σ-continuity of the operators $T(t)$, that $T(t)f \to T(s)f$ as $t \downarrow s$ for every $f \in C_b(E)$, i.e. the orbits $t \mapsto T(t)f$ are right-continuous with respect to σ. In particular, the orbits have enough measurability to define the Laplace transform of a C_b-Feller semigroup by setting

$$\langle R(\lambda)f, \nu\rangle := \int_0^\infty e^{-\lambda t} \langle T(t)f, \nu\rangle \,\mathrm{d}t \tag{A.2}$$

for any $f \in C_b(E)$, $\nu \in \mathscr{M}(E)$ and $\lambda > 0$.

Lemma A.3 *Let $(T(t), t > 0)$ be a C_b-Feller semigroup. Then, for every $\lambda > 0$, Equation A.2 defines an operator $R(\lambda) \in \mathscr{L}(C_b(E), \sigma)$. Moreover, the family $(R(\lambda), \lambda > 0)$ consists of injective operators and satisfies the* resolvent identity

$$R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1)R(\lambda_2)R(\lambda_1)$$

for all $\lambda_1, \lambda_2 > 0$.

Proof By [46, Theorem 6.2] any C_b-Feller semigroup is integrable in the sense of [46, Definition 5.1]. Now the resolvent identity for the operators R_λ follows from [46, Proposition 5.2]. That the operators $R(\lambda)$ are injective is a consequence of [45, Theorem 2.10]. □

As is well known, if $(R(\lambda), \lambda > 0)$ consists of injective operators and satisfies the resolvent identity, then there exists a unique operator A $(= \lambda - R(\lambda)^{-1})$ such that $R(\lambda) = (\lambda - A)^{-1}$.

Definition A.4 Let $(T(t), t > 0)$ be a C_b-Feller semigroup. The *C_b-generator* of $(T(t), t > 0)$ is the unique operator A such that $R(\lambda) = (\lambda - A)^{-1}$ for all $\lambda > 0$, where the operators $R(\lambda)$ are given by Eq. A.2, and its domain is $D(A) := \mathrm{rg} R(\lambda)$, which is independent of $\lambda > 0$.

The above gives an "integral" definition of the (C_b-)generator by means of the Laplace transform of the semigroup. Often, a differential definition of the generator is preferred and we show next that several differential definitions are in fact equivalent to the above. In one of them, we make use of the so-called *strict topology* β_0 on $C_b(E)$. This topology is defined as follows: Let $\mathscr{F}_0(E)$ denote the set of functions $\varphi : E \to \mathbb{R}$ that *vanish at infinity*, i.e. for every $\varepsilon > 0$ there exists a compact set $H \subset E$ with $|\varphi(x)| \leq \varepsilon$ for all $x \in E \setminus H$. Then the *strict topology* β_0 is the locally convex topology generated by the seminorms $\{p_\varphi : \varphi \in \mathscr{F}_0\}$, where $p_\varphi(f) = \|\varphi f\|_\infty$. This topology is consistent with the duality $(C_b(E), \mathscr{M}(E))$, i.e., the dual space $(C_b(E), \beta_0)'$ is $\mathscr{M}(E)$, see [39, Theorem 7.6.3]. In fact, it is the Mackey topology of the dual pair $(C_b(E), \mathscr{M}(E))$, i.e. the finest locally convex topology on $C_b(E)$ that yields $\mathscr{M}(E)$ as a dual space, see [61, Theorem 4.5 and 5.8]. This implies that a kernel operator is automatically also β_0-continuous. By [39, Theorem 2.10.4], β_0 coincides on $\|\cdot\|_\infty$-bounded subsets on $C_b(E)$ with the topology of uniform convergence on compact subsets of E. Thus, condition (iii) in Definition A.2 can be reformulated by saying $T(t)f \to f$ with respect to β_0 as $t \to 0$ for every $f \in C_b(E)$. Taking the β_0-continuity of the operators $T(t)$ into account, it follows that for every $f \in C_b(E)$ the orbit $t \mapsto T(t)f$ is β_0 right-continuous.

Theorem A.5 *Let $(T(t), t > 0)$ be a C_b-Feller semigroup with C_b-generator A. Then for $u, f \in C_b(E)$, the following assertions are equivalent.*

(i) $u \in D(A)$ *and* $Au = f$.
(ii) *For every* $t > 0$ *and* $x \in E$, *we have* $T(t)u(x) - u(x) = \int_0^t T(s)f(x)\,\mathrm{d}s$.
(iii) $\sup\{t^{-1}\|T(t)u - u\|_\infty : t \in (0,1)\} < \infty$ *and* $t^{-1}(T(t)u(x) - u(x)) \to f(x)$ *as* $t \to 0$ *for all* $x \in E$.
(iv) $t^{-1}(T(t)u - u) \to f$ *with respect to* σ *as* $t \to 0$.
(v) $t^{-1}(T(t)u - u) \to f$ *with respect to* β_0 *as* $t \to 0$.
(vi) $\sup\{t^{-1}\|T(t)u - u\|_\infty : t \in (0,1)\} < \infty$ *and* $t^{-1}(T(t)u - u) \to f$ *as* $t \to 0$ *uniformly on compact subsets of* E.

Proof (i) ⇒ (ii). By [46, Proposition 5.7](i), $\langle T(t)u - u, \nu\rangle = \int_0^t \langle T(s)f, \nu\rangle\,\mathrm{d}s$ for all $t > 0$ and $\nu \in \mathscr{M}(E)$. Picking $\nu = \delta_x$, we get (ii).

(ii) $\Rightarrow$ (iii). We have $t^{-1}(T(t)u(x) - u(x)) = t^{-1}\int_0^t T(s)f(x)\,\mathrm{d}s \to f(x)$ as $t \to 0$, by the continuity of $s \mapsto T(s)f(x)$ in 0. Moreover,

$$\|t^{-1}(T(t)u(x) - u(x)\|_\infty \leq t^{-1}\int_0^t \|T(s)f\|_\infty \mathrm{d}s \leq \|f\|_\infty < \infty$$

for all $t > 0$.

(iii) $\Rightarrow$ (iv) follows from the dominated convergence theorem, whereas (iv) $\Rightarrow$ (i) is a consequence of [45, Theorem 2.10], applied with $\tau_{\mathfrak{M}} = \sigma$, which corresponds to choosing $\mathfrak{M}$ as the finite subsets of $Y = \mathscr{M}(E)$.

As β_0 is the Mackey topology of the pair $(C_b(E), \mathscr{M}(E))$, we have $\beta = \tau_{\mathfrak{M}}$ where $\mathfrak{M}$ denotes the collection of all absolutely convex subsets of $Y = \mathscr{M}(E)$ which are $\sigma(\mathscr{M}(E), C_b(E))$-compact. Thus the equivalence (i) $\Leftrightarrow$ (v) also follows from [45, Theorem 2.10], this time applied with $\tau_{\mathfrak{M}} = \beta_0$. The remaining equivalence (v) $\Leftrightarrow$ (vi) follows from the fact that β_0 coincides with the topology of uniform convergence on compact subset of E on $\|\cdot\|_\infty$-bounded subsets of $C_b(E)$ and the already established implications (v) $\Rightarrow$ (i) $\Rightarrow$ (iii). ◻

If $(T(t), t > 0)$ is a C_b-Feller semigroup then, by the β_0-continuity of the operators $T(t)$ and (iii) in Definition A.2, for every $f \in C_b(E)$ the orbit $t \mapsto T(t)f$ is right-continuous with respect to β_0. It is a natural question, whether each orbit is actually β_0-continuous, but, to the best of our knowledge, it is still open. However, if $(T(t), t > 0)$ additionally enjoys the *strong Feller property*, i.e. $T(t)B_b(E) \subset C_b(E)$ for all $t > 0$, then this is indeed the case.

Lemma A.6 *Let $(T(t), t > 0)$ be a C_b-Feller semigroup that enjoys the strong Feller property. Then $(T(t), t > 0)$ has the following additional properties. Here, in parts (b) and (c) we set $T(0) = I$.*

(a) For every $f \in B_b(E)$, the map $(0, \infty) \times E \ni (t, x) \mapsto T(t)f(x)$ is continuous.
(b) For every $f \in C_b(E)$, the map $[0, \infty) \ni t \mapsto T(t)$ is β_0-continuous.
(c) For every $f \in C_b(E)$ and $t_0 \in [0, \infty)$, we have $T_t f \to T_{t_0} f$ as $t \to t_0$ uniformly on compact subsets of E.

Proof (a) follows from [5, Proposition V.2.10]. See also [47, Theorem 3.7], which shows that the continuity assumption in [5] can be weakened to a measurability assumption. It follows from (a) and (iii) in Definition A.2, that for $f \in C_b(E)$ the map $[0, \infty) \times E \ni (t, x) \mapsto T_t f(x)$ is continuous. Now (b) and (c) follow from [45, Theorem 4.4]. ◻

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References

1. Arendt, W., Kunkel, S., Kunze, M.: Diffusion with nonlocal boundary conditions. J. Funct. Anal. **270**(7), 2483–2507 (2016)
2. Baeumer, B., Luks, T., Meerschaert, M.M.: Space-time fractional Dirichlet problems. Math. Nachr. **291**(17–18), 2516–2535 (2018)
3. Barles, G., Chasseigne, E., Georgelin, C., Jakobsen, E.R.: On Neumann type problems for nonlocal equations set in a half space. Trans. Amer. Math. Soc. **366**(9), 4873–4917 (2014)
4. Ben-Ari, I., Pinsky, R.G.: Spectral analysis of a family of second-order elliptic operators with nonlocal boundary condition indexed by a probability measure. J. Funct. Anal. **251**(1), 122–140 (2007)
5. Bliedtner, J., Hansen, W.: Potential theory. Universitext. Springer-Verlag, Berlin. An analytic and probabilistic approach to balayage. (1986)
6. Bobrowski, A.: Concatenation of dishonest Feller processes, exit laws, and limit theorems on graphs (2022)
7. Bogachev, V.I.: Measure theory, vol. I. II. Springer-Verlag, Berlin (2007)
8. Bogdan, K.: The boundary Harnack principle for the fractional Laplacian. Studia Math. **123**(1), 43–80 (1997)
9. Bogdan, K., Burdzy, K., Chen, Z.-Q.: Censored stable processes. Probab. Theory Related Fields **127**(1), 89–152 (2003)
10. Bogdan, K., Byczkowski, T.: Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains. Studia Math. **133**(1), 53–92 (1999)
11. Bogdan, K., Byczkowski, T.: Probabilistic proof of boundary Harnack principle for α-harmonic functions. Potential Anal. **11**(2), 135–156 (1999)
12. Bogdan, K., Grzywny, T., Pietruska-Pałuba, K., Rutkowski, A.: Extension and trace for nonlocal operators. J. Math, Pures Appl (2019)
13. Bogdan, K., Grzywny, T., Ryznar, M.: Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. Ann. Probab. **38**(5), 1901–1923 (2010)
14. Bogdan, K., Grzywny, T., Ryznar, M.: Density and tails of unimodal convolution semigroups. J. Funct. Anal. **266**(6), 3543–3571 (2014)
15. Bogdan, K., Grzywny, T., Ryznar, M.: Barriers, exit time and survival probability for unimodal Lévy processes. Probab. Theory Related Fields **162**(1–2), 155–198 (2015)
16. Bogdan, K., Hansen, W., Jakubowski, T.: Time-dependent Schrödinger perturbations of transition densities. Studia Math. **189**(3), 235–254 (2008)
17. Bogdan, K., Kunze, M.: Stable processes with reflections. in preparation
18. Bogdan, K., Rosiński, J., Serafin, G., Wojciechowski, Ł: Lévy systems and moment formulas for mixed Poisson integrals. In: Stochastic analysis and related topics, volume 72 of Progr. Probab, pp. 139–164. Birkhäuser/Springer, Cham (2017)
19. Bogdan, K., Sydor, S.: On nonlocal perturbations of integral kernels. In Semigroups of operators—theory and applications, volume 113 of Springer Proc. Math. Stat. pp 27–42. Springer, Cham (2015)
20. Böttcher, B., Schilling, R., Wang, J.: Lévy matters. III, volume 2099 of Lecture Notes in Mathematics. Springer, Cham, 2013. Lévy-type processes: construction, approximation and sample path properties, With a short biography of Paul Lévy by Jean Jacod
21. Chaumont, L., Pantí, H., Rivero, V.: The Lamperti representation of real-valued self-similar Markov processes. Bernoulli **19**(5B), 2494–2523 (2013)
22. Chen, Z.-Q., Song, R.: Intrinsic ultracontractivity and conditional gauge for symmetric stable processes. J. Funct. Anal. **150**(1), 204–239 (1997)
23. Chen, Z.-Q., Song, R.: Conditional gauge theorem for non-local Feynman-Kac transforms. Probab. Theory Related Fields **125**(1), 45–72 (2003)

24. Chung, K.L.: Doubly-Feller process with multiplicative functional. In Seminar on stochastic processes, 1985 (Gainesville, Fla., 1985), volume 12 of Progr. Probab. Statist. pp 63–78. Birkhäuser Boston, Boston, MA (1986)
25. Chung, K.L., Walsh, J.B.: Markov processes, Brownian motion, and time symmetry, volume 249 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, New York, second edition (2005)
26. Chung, K.L., Zhao, Z.X.: From Brownian motion to Schrödinger's equation, volume 312 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin (1995)
27. Da Prato, G., Zabczyk, J.: Ergodicity for infinite-dimensional systems, volume 229 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (1996)
28. Dipierro, S., Ros-Oton, X., Valdinoci, E.: Nonlocal problems with Neumann boundary conditions. Rev. Mat. Iberoam. **33**(2), 377–416 (2017)
29. Ethier, S.N., Kurtz, T.G.: Markov processes. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York. Characterization and convergence. (1986)
30. Evans, M.R., Majumdar, S.N., Schehr, G.: Stochastic resetting and applications. J. Phys. A **53**(19), 193001, 67 (2020)
31. Feller, W.: Diffusion processes in one dimension. Trans. Amer. Math. Soc. **77**, 1–31 (1954)
32. Fitzsimmons, P.J.: On the existence of recurrent extensions of self-similar Markov processes. Electron. Comm. Probab. **11**, 230–241 (2006)
33. Galakhov, E.I., Skubachevskiĭ, A.L.: On Feller semigroups generated by elliptic operators with integro-differential boundary conditions. J. Differential Equat. **176**(2), 315–355 (2001)
34. Garbaczewski, P., Żaba, M.: Lévy processes in bounded domains: path-wise reflection scenarios and signatures of confinement. J. Phys. A **55**(30), Paper No. 305005, 26 (2022)
35. Hairer, M.: Convergence of Markov processes. Minicourse available at http://www.hairer.org/notes/Convergence.pdf, (2021)
36. Ikeda, N., Nagasawa, M., Watanabe, S.: A construction of Markov processes by piecing out. Proc. Japan Acad. **42**, 370–375 (1966)
37. Iksanov, A., Pilipenko, A.: On a skew stable Lévy process. Stochastic Process. Appl. **156**, 44–68 (2023)
38. Jacob, N.: Pseudo differential operators and Markov processes, vol. I. Imperial College Press, London. Fourier analysis and semigroups (2001)
39. Jarchow, H.: Locally convex spaces. Teubner, B.G., Stuttgart, (1981). Mathematische Leitfäden. [Mathematical Textbooks]
40. Kim, P., Song, R., Vondraček, Z.: Positive self-similar Markov processes obtained by resurrection. Stochastic Process. Appl. **156**, 379–420 (2023)
41. Kim, P., Song, R., Vondraček, Z.: Potential theory of dirichlet forms with jump kernels blowing up at the boundary (2022)
42. Komorowski, T.: Asymptotic periodicity of some stochastically perturbed dynamical systems. Ann. Inst. H. Poincaré Probab. Statist. **28**(2), 165–178 (1992)
43. Kreĭn, S.G.: Linear equations in Banach spaces. Birkhäuser, Boston, Mass.,: Translated from the Russian by A. Gohberg, Iacob, With an introduction by I (1982)
44. Kulik, A.: Introduction to ergodic rates for Markov chains and processes, volume 2 of Lectures in Pure and Applied Mathematics. Potsdam University Press, Potsdam. With applications to limit theorems (2015)
45. Kunze, M.: Continuity and equicontinuity of semigroups on norming dual pairs. Semigroup Forum **79**(3), 540–560 (2009)
46. Kunze, M.: A Pettis-type integral and applications to transition semigroups. Czechoslovak Math. J. **61**(136)(2), 437–459 (2011)
47. Kunze, M.: Diffusion with nonlocal Dirichlet boundary conditions on unbounded domains. Studia Math. **253**(1), 1–38 (2020)
48. Kwaśnicki, M.: Ten equivalent definitions of the fractional Laplace operator. Fract. Calc. Appl. Anal. **20**(1), 7–51 (2017)
49. Lasota, A.: From fractals to stochastic differential equations. In: Chaos–the interplay between stochastic and deterministic behaviour (Karpacz. 1995), volume 457 of Lecture Notes in Phys, pp 235–255. Springer, Berlin (1995)
50. Lasota, A., Mackey, M.C.: Chaos, fractals, and noise, volume 97 of Applied Mathematical Sciences. Springer-Verlag, New York, second edition. Stochastic aspects of dynamics (1994)
51. Lasota, A., Yorke, J.A.: Lower bound technique for Markov operators and iterated function systems. Random Comput. Dynam. **2**(1), 41–77 (1994)
52. Meyer, P. A.: Renaissance, recollements, m'elanges, ralentissement de processus de Markov. Ann. Inst. Fourier (Grenoble) **25**(3-4):xxiii, 465–497 (1975)

53. Pantí, H., Pardo, J.C., Rivero, V.M.: Recurrent extensions of real-valued self-similar Markov processes. Potential Anal. **53**(3), 899–920 (2020)
54. Protter, P.E.: Stochastic integration and differential equations, volume 21 of Stochastic Modelling and Applied Probability. Springer-Verlag, Berlin. Second edition. Version 2.1, Corrected third printing (2005)
55. Revuz, D.: Markov chains. North-Holland Publishing Co., Amsterdam. North-Holland Mathematical Library, **11**, (1975)
56. Rivero, V.: Recurrent extensions of self-similar Markov processes and Cramér's condition. II. Bernoulli **13**(4), 1053–1070 (2007)
57. Sato, K.: Lévy processes and infinitely divisible distributions, volume 68 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, Translated from the 1990 Japanese original, Revised by the author (1999)
58. Schaefer, H.H.: Banach lattices and positive operators. Die Grundlehren der mathematischen Wissenschaften, Band 215. Springer-Verlag, New York-Heidelberg, (1974)
59. Schilling, R.L.: Conservativeness and extensions of Feller semigroups. Positivity **2**(3), 239–256 (1998)
60. Schilling, R.L.: Measures, integrals and martingales. Cambridge University Press, Cambridge, second edition (2017)
61. Sentilles, F.D.: Bounded continuous functions on a completely regular space. Trans. Amer. Math. Soc. **168**, 311–336 (1972)
62. Sharpe, M.: General theory of Markov processes, volume 133 of Pure and Applied Mathematics. Academic Press, Inc., Boston, MA (1988)
63. Stanislavsky, A.A., Weron, A.: Subdiffusive search with home returns via stochastic resetting: a subordination scheme approach. J. Phys. A **55**(7), Paper No. 074004, 15 (2022)
64. Taira, K.: Semigroups, boundary value problems and Markov processes. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2004)
65. Vondraček, Z.: A probabilistic approach to a non-local quadratic form and its connection to the Neumann boundary condition problem. Math. Nachr. **294**(1), 177–194 (2021)
66. Werner, F.: Concatenation and pasting of right processes. Electron. J. Probab. **26**, Paper No. 50, 21 (2021)