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ORIGINAL PAPER

Spherical Two-Distance Sets and Eigenvalues of Signed Graphs

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Abstract

We study the problem of determining the maximum size of a spherical two-distance set with two fixed angles (one acute and one obtuse) in high dimensions. Let $N_{\alpha,\beta}(d)$ denote the maximum number of unit vectors in $\mathbb{R}^d$ where all pairwise inner products lie in $\{\alpha, \beta\}$. For fixed $-1 \le \beta < 0 \le \alpha < 1$, we propose a conjecture for the limit of $N_{\alpha,\beta}(d)/d$ as $d \to \infty$ in terms of eigenvalue multiplicities of signed graphs. We determine this limit when $\alpha + 2\beta < 0$ or $(1-\alpha)/(\alpha-\beta) \in \{1, \sqrt{2}, \sqrt{3}\}$.
Our work builds on our recent resolution of the problem in the case of $\alpha = -\beta$ (corresponding to equiangular lines). It is the first determination of $\lim_{d\to\infty} N_{\alpha,\beta}(d)/d$ for any nontrivial fixed values of α and β outside of the equiangular lines setting.

Keywords Spherical two-distance set · Eigenvalue multiplicity · Signed graph

Mathematics Subject Classification 52C35 · 05C50 · 05D10

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1 Introduction

A set of unit vectors in $\mathbb{R}^d$ is a *spherical two-distance set* if the inner products of distinct vectors only take two values. The problem of determining the maximum size of spherical two-distance sets is a deep and natural problem in discrete geometry. Some of the earliest results in this area date to the seminal work of Delsarte et al. [4]. They prove that a spherical two-distance set in $\mathbb{R}^d$ has size at most $\frac{1}{2}d(d+3)$. This bound is close to the truth, as taking the $\frac{1}{2}d(d+1)$ midpoints on the edges of a regular simplex form a spherical two-distance set in $\mathbb{R}^d$. Recently Glazyrin and Yu [6] determined that the maximum size of spherical two-distance sets in $\mathbb{R}^d$ is indeed $\frac{1}{2}d(d+1)$ whenever $d \geq 7$ and $d+3$ is not an odd perfect square; see [2, 14, 17] for results in many small dimensions.

Given a set $A \subset [-1, 1)$, a *spherical A-code* is a set S of unit vectors in $\mathbb{R}^d$ where $\langle x, y\rangle \in A$ for all distinct $x, y \in S$. We write $N_A(d)$ the maximum size of a spherical A-code in $\mathbb{R}^d$. In this paper, we are primarily interested in the case $A = \{\alpha, \beta\}$ for fixed $-1 \leq \beta < \alpha < 1$ and large d, in which case we write $N_{\alpha,\beta}(d)$ instead of $N_{\{\alpha,\beta\}}(d)$.

Let us briefly mention some early developments on this problem. The special case $\alpha = -\beta$ corresponds to *equiangular lines*, whose study in the setting of fixed angle in high dimensions began with the work of Lemmens and Seidel [12]. For spherical two-distance sets with fixed angles, Neumaier [15, Corollary 5] showed that $N_{\alpha,\beta}(d) \leq 2d+1$ unless $(1-\alpha)/(\alpha-\beta)$ is an integer. Furthermore, a result of Larman et al. [11] implies the growth rate $N_{\alpha,\beta}(d) = \Theta_{\alpha,\beta}(d^2)$ for all $0 \leq \beta < \alpha < 1$ such that $(1-\alpha)/(\alpha-\beta)$ is an integer.[1] The regime $-1 \leq \beta < \alpha < 0$ is less interesting, as an easy argument shows that $N_{[-1,\alpha]}(d) \leq 1 - 1/\alpha$ for all $\alpha < 0$.

Recently work [1, 3, 9] culminated in a solution [10] to the problem of determining the maximum number of equiangular lines with fixed angles in high dimensions. The papers [1, 3] also address the more general problem of estimating $N_{[-1,\beta]\cup\{\alpha_1,\ldots,\alpha_k\}}(d)$ for fixed $\beta < 0 < \alpha_1 < \cdots < \alpha_k$. In particular, Bukh [3] showed that $N_{[-1,\beta]\cup\{\alpha\}}(d) = O_\beta(d)$, in sharp contrast to the quadratic dependence in dimension without angle restrictions. Significant progress was made by Balla et al. [1], whose results in particular imply the bound $N_{\alpha,\beta}(d) \leq 2(1-\alpha/\beta+o(1))d$. More generally, it was conjectured in [3] and proved in [1, Theorem 1.4] that $N_{[-1,\beta]\cup\{\alpha_1,\ldots,\alpha_k\}} = O_{k,\beta}(d^k)$, and that there exist choices of $\alpha_1, \ldots, \alpha_k, \beta$ for which this upper bound is tight up to a constant factor. Here subscripts in the asymptotic notation indicate that hidden constants may depend on these parameters.

We focus our attention on the goal of sharpening the above results for spherical two-distance sets and obtaining tight asymptotics for their maximum sizes.

Problem 1.1 *Determine, for fixed $-1 \leq \beta < 0 \leq \alpha < 1$, and large d, the maximum number, denoted $N_{\alpha,\beta}(d)$, of unit vectors in $\mathbb{R}^d$ whose pairwise inner products lie in $\{\alpha, \beta\}$. In particular determine the limit of $N_{\alpha,\beta}(d)/d$ as $d \to \infty$.*

[1] Using Wilson's deep result [16] on the existence of balanced incomplete block designs, Larman et al. [11, Theorem 3] constructed a spherical $\{0, 1/(\lambda+1)\}$-code $C_\lambda(d)$ of size $\Theta_\lambda(d^2)$ in $\mathbb{R}^d$ for any positive integer λ, from which one constructs a spherical $\{\alpha,\beta\}$-code $\{\sqrt{1-\beta}(\boldsymbol{v}, \sqrt{\beta/(1-\beta)})\colon \boldsymbol{v} \in C_\lambda(d)\}$ of size $\Theta_\lambda(d^2)$ in $\mathbb{R}^{d+1}$ for every $0 \leq \beta < \alpha$ with $\lambda = (1-\alpha)/(\alpha-\beta)$ a positive integer.

We recently solved Problem 1.1 in the case of equiangular lines [10] where $\beta = -\alpha$. To state the result, we need the following spectral graph quantity, introduced in [9].

Definition 1.2 The *spectral radius order*, denoted $k(\lambda)$, of a real number $\lambda > 0$ is the smallest integer k so that there exists a k-vertex graph G whose spectral radius $\lambda_1(G)$ is exactly λ. Set $k(\lambda) = \infty$ if no such graph exists. (When we talk about the spectral radius or eigenvalues of a graph we always refer to its adjacency matrix.)

Theorem 1.3 (Equiangular lines with a fixed angle [10]) Fix $\alpha \in (0, 1)$. Let $\lambda = (1-\alpha)/(2\alpha)$. For all sufficiently large $d > d_0(\alpha)$,

$$N_{\alpha,-\alpha}(d) = \begin{cases} \left\lfloor \dfrac{k(\lambda)(d-1)}{k(\lambda)-1} \right\rfloor & \text{if } k(\lambda) < \infty, \\ d + o(d) & \text{otherwise.} \end{cases}$$

Let us recap some key steps in the proof of the upper bound on $N_{\alpha,-\alpha}(d)$ in Theorem 1.3. We will build on this framework.

Given a spherical $\{\pm\alpha\}$-code S, we consider the *associated graph* G with vertex set S, where $x, y \in S$ are adjacent in G if $\langle x, y\rangle = -\alpha$. We are allowed to replace any $x \in S$ by $-x$ without changing the equiangular lines configuration. An argument introduced in [1] reduces the problem to bounded degree graphs.

Theorem 1.4 ([1] and [10, Theorem 2.1]) For every $\alpha \in (0, 1)$, there exists Δ depending only on α, such that given any spherical $\{\pm\alpha\}$-code S in $\mathbb{R}^n$, one can replace some subset of vectors in S by their negations so that the associated graph G (as defined above) has maximum degree at most Δ.

The problem of bounding the size of S is related to the multiplicity of $(1-\alpha)/(2\alpha)$ as the second largest eigenvalue of the adjacency matrix of G. A crucial contribution of [10] is that every connected bounded degree graph has sublinear second eigenvalue multiplicity. More generally, we have the following. (See Definition 1.8 below for the precise definition of j-th eigenvalue multiplicity.)

Theorem 1.5 ([10, Theorem 2.2]) For every j and Δ, there is a constant $C = C(\Delta, j)$ so that every connected n-vertex graph with maximum degree at most Δ has j-th eigenvalue multiplicity at most $Cn/\log\log n$.

Turning to spherical two-distance sets, given a spherical $\{\alpha, \beta\}$-code S (with $\beta < 0 \le \alpha$ as always throughout this paper), we define its *associated graph* G to have vertex set S and where $x, y \in S$ are adjacent in G if $\langle x, y\rangle = \beta$. Unlike for equiangular lines, here we are no longer allowed to negate a subset of vectors in a spherical $\{\alpha, \beta\}$-code. Instead, we show that G is very close to a complete p-partite graph. Here p is a specific constant, with the equiangular lines problem corresponding to $p = 2$.

Definition 1.6 A graph G is a Δ-*modification* of another graph H on the same vertex set if the symmetric difference of G and H has maximum degree at most Δ.

Theorem 1.7 *For every $-1 \le \beta < 0 \le \alpha < 1$, there exists Δ depending only on α and β such that for every spherical $\{\alpha, \beta\}$-code, its associated graph G (as defined above), after removing at most Δ vertices, is a Δ-modification of a complete p-partite graph, where $p = \lfloor -\alpha/\beta \rfloor + 1$.*

Remark We allow empty parts in a complete p-partite graph. In particular, a complete t-partite graph is always a complete p-partite graph for $t \le p$.

It will be helpful to study such graphs using the language of signed graphs.

Definition 1.8 A *signed graph* is a graph whose edges are each labeled by $+$ or $-$. Throughout the paper we decorate variables for signed graphs with the $\pm$ superscript. The *signed adjacency matrix* $A_{G^\pm}$ of a signed graph $G^\pm$ on n vertices is the $n \times n$ matrix whose (i, j)-th entry is 1 if ij is a positive edge, and -1 if ij is a negative edge, and 0 otherwise. We denote the eigenvalues of $A_{G^\pm}$ by $\lambda_1(G^\pm) \ge \lambda_2(G^\pm) \ge \cdots \ge \lambda_n(G^\pm)$. We write

$$\mathrm{mult}(\lambda, G^\pm) = \left|\{i : \lambda_i(G^\pm) = \lambda\}\right|$$

for the the multiplicity of λ as an eigenvalue of $G^\pm$. The j-th *eigenvalue multiplicity* of $G^\pm$ is defined to be $\mathrm{mult}(\lambda_j(G^\pm), G^\pm)$. We use $|G|$ and $\left|G^\pm\right|$ to denote the number of vertices in the graph.

Given a Δ-modification G of a complete p-partite graph K, we study the signed graph $G^\pm$ defined by $A_{G^\pm} = A_G - A_K$. The growth rate of $N_{\alpha,\beta}(d)$ is related to the eigenvalue multiplicity of $G^\pm$. We introduce the following parameter generalizing the spectral radius order $k(\lambda)$ for signed graphs.

Definition 1.9 A *valid p-coloring* of a signed graph $G^\pm$ is a coloring of the vertices using p colors such that the endpoints of every negative edge are colored using distinct colors, and the endpoints of every positive edge are colored using identical colors. (See Fig. 1 for an example.) The *chromatic number* $\chi(G^\pm)$ of a signed graph $G^\pm$ is the smallest p for which $G^\pm$ has a valid p-coloring. If $G^\pm$ does not have a valid p-coloring for any p, we write $\chi(G^\pm) = \infty$.

Definition 1.10 Given $\lambda > 0$ and $p \in \mathbb{N}$, define the parameter

$$k_p(\lambda) = \inf\left\{\frac{\left|G^\pm\right|}{\mathrm{mult}(\lambda, G^\pm)} : \chi(G^\pm) \le p \text{ and } \lambda_1(G^\pm) = \lambda\right\}.$$

We say that $k_p(\lambda)$ is *achievable* if it is finite and the infimum can be attained.

In the definition of $k_p(G^\pm)$, it is enough to consider connected $G^\pm$, since the eigenvalues of $G^\pm$ are given by the union of the the eigenvalues of its connected components.

If $\chi(G^\pm) \le 2$, then the signed graph $G^\pm$ and its underlying graph G have the same eigenvalues (including multiplicities), since the signed adjacency matrix of $G^\pm$ can be

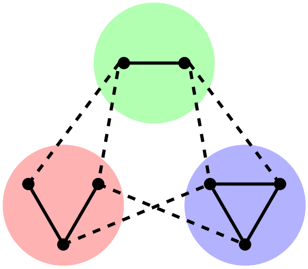


Fig. 1 A valid 3-coloring of a signed graph. Throughout this paper, the positive edges are represented by solid segments and the negative edges are represented by dashed segments

obtained by conjugating the adjacency matrix of G by a $\{\pm 1\}$-valued diagonal matrix. By the Perron–Frobenius theorem, the top eigenvalue of a connected unsigned graph has multiplicity one. Thus,

$$k_1(\lambda) = k_2(\lambda) = k(\lambda) \qquad \text{for all } \lambda > 0.$$

However the behavior of $k_p(\lambda)$ is far more mysterious when $p \geq 3$. We do not know any general method of estimating or certifying values of $k_p(\lambda)$. Also, it is not even clear whether the infimum in the definition of $k_p(\lambda)$ can always be attained whenever $k_p(\lambda)$ is finite.

Generalizing the construction in [9] relating equiangular lines to $k(\lambda)$, we can obtain a lower bound on $\lim_{d\to\infty} N_{\alpha,\beta}(d)/d$ (see Proposition 2.2). Our main conjecture, below, says that this lower bound is sharp.

Conjecture 1.11 *Fix* $-1 \leq \beta < 0 \leq \alpha < 1$. *Set* $\lambda = (1-\alpha)/(\alpha-\beta)$ *and* $p = \lfloor -\alpha/\beta \rfloor + 1$. *Then*

$$\lim_{d\to\infty} \frac{N_{\alpha,\beta}(d)}{d} = \begin{cases} \dfrac{k_p(\lambda)}{k_p(\lambda)-1} & \text{if } k_p(\lambda) < \infty, \\ 1 & \text{otherwise.} \end{cases}$$

We see above that the parameters

$$\lambda = \frac{1-\alpha}{\alpha-\beta} \qquad \text{and} \qquad p = \left\lfloor -\frac{\alpha}{\beta} \right\rfloor + 1$$

appear to play important roles in the problem. These two parameters λ and p conjecturally govern the asymptotic behavior of $N_{\alpha,\beta}(d)$. Our main theorem below establishes Conjecture 1.11 for $p \leq 2$, as well as for $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$. This is the first time that some $\lim_{d\to\infty} N_{\alpha,\beta}(d)/d$ is determined outside of the equiangular lines setting ($\alpha = -\beta$).

Theorem 1.12 *Fix* $-1 \leq \beta < 0 \leq \alpha < 1$. *Set* $\lambda = (1-\alpha)/(\alpha-\beta)$ *and* $p = \lfloor -\alpha/\beta \rfloor + 1$.

(a) If $p \le 2$, then the maximum size $N_{\alpha,\beta}(d)$ of a spherical $\{\alpha,\beta\}$-code in $\mathbb{R}^d$ satisfies

$$N_{\alpha,\beta}(d) = \begin{cases} \dfrac{k(\lambda)d}{k(\lambda)-1} + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ d + o(d) & \text{otherwise.} \end{cases}$$

(b) If $\lambda = 1$ and $p \ge 2$, then $k_p(1) = p/(p-1)$ and $N_{\alpha,\beta}(d) = pd + O_{\alpha,\beta}(1)$.
(c) If $\lambda = \sqrt{3}$ and $p = 3$, then $k_3(\sqrt{3}) = 7/3$ and $N_{\alpha,\beta}(d) = 7d/4 + O_{\alpha,\beta}(1)$.
(d) If $\lambda \in \{\sqrt{2}, \sqrt{3}\}$ and $p \ge \lambda^2 + 1$, then $k_p(\lambda) = 2$ and $N_{\alpha,\beta}(d) = 2d + O_{\alpha,\beta}(1)$.

Moreover, $k_p(\lambda)$ is achievable for every $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$ and $p \in \mathbb{N}$.

Remark The conditions on λ and p in Theorem 1.12 can be directly translated to ones on α and β. The condition $p \le 2$ in (a) amounts to $\alpha + 2\beta < 0$, which includes the special case $\alpha = -\beta$ for equiangular lines. The conditions in both (b) and (d) amount to $(\lambda+1)\alpha - \lambda\beta = 1$ and $\lambda/(\lambda^2+\lambda+1) \le \alpha < 1/(\lambda+1)$, where $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$. For example, $(\alpha, \beta) = (2/5, -1/5)$ satisfies the last two conditions for $\lambda = 1$, yielding $N_{2/5,-1/5}(d) = 3d + O(1)$. It is worth contrasting the last example to the universal equiangular lines bound $N_{\alpha,-\alpha}(d) \le 2d + O_\alpha(1)$ for all fixed $\alpha > 0$ (implied by Theorem 1.3, but proved initially in [1]). Lastly the condition in (c) amounts to $(\sqrt{2}+1)\alpha - \sqrt{2}\beta = 1$ and $2/(3\sqrt{2}+2) \le \alpha < 3/(4\sqrt{2}+3)$.

We also prove a general upper bound on $N_{\alpha,\beta}(d)$, though it is not expected to be tight except for special values (e.g., it implies Theorem 1.12(a)(b)).

Theorem 1.13 *Fix $-1 \le \beta < 0 \le \alpha < 1$. Set $\lambda = (1-\alpha)/(\alpha-\beta)$ and $p = \lfloor -\alpha/\beta \rfloor + 1$ and $q = \max\{1, p/2\}$. Then*

$$N_{\alpha,\beta}(d) \le \begin{cases} \dfrac{qk(\lambda)d}{k(\lambda)-1} + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ qd + o(d) & \text{otherwise.} \end{cases}$$

Our proof of Theorem 1.12 indeed confirms Conjecture 1.11 in all the solved cases, namely when $p \le 2$ or $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$. We employ a number of different methods for bounding eigenvalue multiplicities in signed graphs in the different parts of Theorem 1.12:

- For (a) and (b), we apply the sublinear bound on eigenvalue multiplicity of bounded degree unsigned graphs (Theorem 1.5 above; see Sect. 4).
- For (c), we develop a forbidden induced subgraph framework (see Sect. 5), and we apply a careful third moment and triangle counting argument (see Sect. 6).
- For (d) we apply an algebraic degree argument (see Sect. 7). Additionally, we confirm Conjecture 1.11 for all algebraic integers λ whose degree equals $k(\lambda)$ (see the end of Sect. 7).

Remark After this work is completed, building on our forbidden induced subgraph framework, Jiang and Polyanskii [8] proved Conjecture 1.11 for every $\lambda < \lambda^*$, where $\lambda^* = \beta^{1/2} + \beta^{-1/2} \approx 2.01980$ and β is the unique real root of $x^3 = x + 1$.

A major obstacle to completely settling Conjecture 1.11 is that bounded degree signed graphs may have linear top eigenvalue multiplicity.

Theorem 1.14 *For every $n \geq 3$, there is a connected signed graph with $6n$ vertices, maximum degree 5, and chromatic number 3, such that its largest eigenvalue appears with multiplicity n.*

The rest of the paper is organized as follows. In Sect. 2, we explain the connection with spherical two-distance sets and the spectral theory of signed graphs, and further proves a lower bound on $N_{\alpha,\beta}(d)$. In Sect. 3 we prove the structural result, Theorem 1.7. In Sect. 4 we prove Theorem 1.12(a), Theorem 1.12(b), and Theorem 1.13 using Theorem 1.5. In Sect. 5 we develop a forbidden induced subgraph framework to bound $N_{\alpha,\beta}(d)$ from above. In Sect. 6 we prove Theorem 1.12(c) via a third moment argument under the forbidden induced subgraph framework. In Sect. 7 we prove Theorem 1.12(d) via an algebraic argument. In Sect. 8 we give two constructions related to Theorem 1.14.

2 Connection to Spectral Theory of Signed Graphs

The spherical two-distance set problem has the following equivalent spectral graph theoretic formulation. Here $A \succeq 0$ means that A is positive semidefinite.

Lemma 2.1 *Let $-1 \leq \beta < \alpha < 1$. Set $\lambda = (1-\alpha)/(\alpha-\beta)$ and $\mu = \alpha/(\alpha-\beta)$. There exists a spherical $\{\alpha, \beta\}$-code of size N in $\mathbb{R}^d$ if and only if there exists a graph G on N vertices satisfying*

$$\lambda I - A_G + \mu J \succeq 0 \quad \textit{and} \quad \operatorname{rank}(\lambda I - A_G + \mu J) \leq d.$$

Proof For a spherical $\{\alpha, \beta\}$-code $\{v_1, \dots, v_N\}$ in $\mathbb{R}^d$, let G be the *associated graph* on vertex set $\{1, \dots, N\}$, where ij is an edge whenever $\langle v_i, v_j\rangle = \beta$. The Gram matrix $M = (\langle v_i, v_j\rangle)_{i,j}$ has 1's on its diagonal and α, β everywhere else, so it equals $(1-\alpha)I - (\alpha-\beta)A_G + \alpha J$, where I is the identity matrix, J the all-ones matrix, and A_G the adjacency matrix of G. We have $M/(\alpha-\beta) = \lambda I - A_G + \mu J$, where $\lambda = (1-\alpha)/(\alpha-\beta)$ and $\mu = \alpha/(\alpha-\beta)$. Since the Gram matrix M is positive semidefinite and has rank at most d, the same holds for $\lambda I - A_G + \mu J$.

Conversely, for every G, λ and μ for which $\lambda I - A_G + \mu J$ is positive semidefinite and has rank d, there exists a corresponding configuration of N unit vectors in $\mathbb{R}^d$, with pairwise inner products in $\{\alpha, \beta\}$. □

We are now ready to establish a lower bound on $N_{\alpha,\beta}(d)$ using Lemma 2.1.

Proposition 2.2 *Fix $-1 \leq \beta < 0 \leq \alpha < 1$. Then $N_{\alpha,\beta}(d) \geq d$ for every positive integer d. Moreover if $k_p(\lambda) < \infty$, where $\lambda = (1-\alpha)/(\alpha-\beta)$ and $p = \lfloor -\alpha/\beta \rfloor + 1$,* *then*

$$N_{\alpha,\beta}(d) \geq \begin{cases} \dfrac{k_p(\lambda)d}{k_p(\lambda)-1} - O_{\alpha,\beta}(1) & \text{if } k_p(\lambda) \text{ is achievable,} \\ \dfrac{k_p(\lambda)d}{k_p(\lambda)-1} - o(d) & \text{otherwise.} \end{cases}$$

Proof Let $\mu = \alpha/(\alpha-\beta)$. Take G to be d-vertex graph with no edges, so that $A_G = 0$ and $\lambda I - A_G + \mu J$ is positive semidefinite and has rank at most d. So $N_{\alpha,\beta}(d) \geq d$ by Lemma 2.1. In fact, the spherical two-distance set constructed here forms a regular $(d-1)$-simplex.

Hereafter assume that $k_p(\lambda) < \infty$. We first construct, for every signed graph $G^{\pm}$ with $\chi(G^{\pm}) \leq p$ and $\lambda_1(G^{\pm}) = \lambda$, a spherical $\{\alpha, \beta\}$-code of size $|G^{\pm}|$ in dimension $|G^{\pm}| - \text{mult}(\lambda, G^{\pm}) + p$. Let $V_1, \dots, V_p$ be the color classes of a valid p-coloring. Consider the unsigned graph G obtained from taking the symmetric difference between the underlying graph of $G^{\pm}$ and the complete p-partite graph with parts $V_1, \dots, V_p$. The adjacency matrix of G is related to the signed adjacency matrix of $G^{\pm}$ by

$$A_G = A_{G^{\pm}} + A_K,$$

where K is the complete p-partite graph with parts $V_1, \dots, V_p$. Therefore,

$$\lambda I - A_G + \mu J = (\lambda I - A_{G^{\pm}}) + (\mu J - A_K).$$

We have $\lambda I - A_{G^{\pm}} \succeq 0$ since $\lambda_1(G^{\pm}) = \lambda$. We now note that $\mu J - A_K$ is positive semidefinite. Indeed, for every $\boldsymbol{x} \in \mathbb{R}^{V(G^{\pm})}$, we set $s_i = \sum_{v \in V_i} \boldsymbol{x}_v$ for each $i \in \{1, \dots, p\}$, and we see

$$\boldsymbol{x}^{\mathsf{T}}(\mu J - A_K)\boldsymbol{x} = \mu \left(\sum_i s_i\right)^2 - \sum_{i \neq j} s_i s_j = \mu \sum_i s_i^2 - (1-\mu) \sum_{i \neq j} s_i s_j.$$

Because $\sum_{i \neq j} s_i s_j \leq (p-1) \sum_i s_i^2$ and $p \leq 1/(1-\mu)$, we conclude that

$$\boldsymbol{x}^{\mathsf{T}}(\mu J - A_K)\boldsymbol{x} \geq \left(\mu - (1-\mu)\left(\frac{1}{1-\mu} - 1\right)\right) \sum_i s_i^2 = 0.$$

Therefore $\mu J - A_K \succeq 0$, and so $\lambda I - A_G + \mu J \succeq 0$. We conclude by Lemma 2.1 that there exists a spherical $\{\alpha, \beta\}$-code of size $|G^{\pm}|$ in $\mathbb{R}^d$, where

$$\begin{aligned} d &= \text{rank}(\lambda I - A_G + \mu J) \leq \text{rank}\,(\lambda I - A_{G^{\pm}}) + \text{rank}(\mu J - A_K) \\ &\leq |G^{\pm}| - \text{mult}(\lambda, G^{\pm}) + p. \end{aligned}$$

Now fix an arbitrary $\varepsilon > 0$. Take a signed graph $G_{\varepsilon}^{\pm}$ such that $|G_{\varepsilon}^{\pm}|/\text{mult}(\lambda, G_{\varepsilon}^{\pm}) \leq k_p(\lambda) + \varepsilon$, $\chi(G_{\varepsilon}^{\pm}) \leq p$, and $\lambda_1(G_{\varepsilon}^{\pm}) = \lambda$. For each positive integer ℓ, denote by $\ell G_{\varepsilon}^{\pm}$

the disjoint union of ℓ copies of $G_\varepsilon^\pm$. We have $\left|\ell G_\varepsilon^\pm\right| = \ell \left|G_\varepsilon^\pm\right|$, $\operatorname{mult}(\lambda, \ell G^\pm) = \ell \operatorname{mult}(\lambda, G^\pm)$, $\chi(\ell G_\varepsilon^\pm) = \chi(G_\varepsilon^\pm) \le p$ and $\lambda_1(\ell G_\varepsilon^\pm) = \lambda_1(G_\varepsilon^\pm) = \lambda$. Thus we can apply the above construction to $G^\pm = \ell G_\varepsilon^\pm$ to obtain a spherical $\{\alpha, \beta\}$-code of size $\ell \left|G_\varepsilon^\pm\right|$ in dimension $\ell(\left|G_\varepsilon^\pm\right| - \operatorname{mult}(\lambda, G_\varepsilon^\pm)) + p$. We conclude that

$$N_{\alpha,\beta}(d) \ge \left|G_\varepsilon^\pm\right| \left\lfloor \frac{d-p}{\left|G_\varepsilon^\pm\right| - \operatorname{mult}(\lambda, G_\varepsilon^\pm)} \right\rfloor \ge \frac{d}{1 - \operatorname{mult}(\lambda, G_\varepsilon^\pm)/\left|G_\varepsilon^\pm\right|} - O_{\alpha,\beta,\varepsilon}(1)$$
$$\ge \frac{d}{1 - 1/(k_p(\lambda)+\varepsilon)} - O_{\alpha,\beta,\varepsilon}(1) = \frac{(k_p(\lambda)+\varepsilon)d}{k_p(\lambda) - 1 + \varepsilon} - O_{\alpha,\beta,\varepsilon}(1).$$

Finally notice that when $k_p(\lambda)$ is achievable, we can take $\varepsilon = 0$ in the above argument. □

3 Structure of the Associated Graph

In this section we prove Theorem 1.7, which gives a structure characterization of graphs that can arise from a spherical two-distance set. To that end, we introduce the following notation.

Definition 3.1 Given a graph G, for sets $Y \subseteq X \subseteq V(G)$, define $C_X(Y)$ to be the set of vertices in $V(G) \setminus X$ that are adjacent to all vertices in Y and not adjacent to any vertices in $X \setminus Y$, and for a set $X \subseteq V(G)$ and $\Delta \in \mathbb{N}$, define

$$C_{X,\Delta} = \bigcup_{Y \subseteq X :\, |Y| \le \Delta} C_X(Y) \quad \text{and} \quad C_{X,-\Delta} = \bigcup_{Y \subseteq X :\, |X \setminus Y| \le \Delta} C_X(Y).$$

We now present a series of structural lemmas leading to the proof of Theorem 1.7.

Lemma 3.2 *For every $\lambda > 0$ and $\mu \in (0,1)$, there exist $\Delta \in \mathbb{N}$ and $L_0 \in \mathbb{N}$ such that for every graph G that satisfies $\lambda I - A_G + \mu J \succeq 0$ the following holds.*

(a) Neither of the following is an induced subgraph of G:

- *(a1) The complete graph K_Δ;*
- *(a2) The complete $(p+1)$-partite graph $K_{\Delta,\dots,\Delta}$, where $p = \lfloor 1/(1-\mu) \rfloor$.*

(b) For every independent set X of size L in G, if $L \ge L_0$, then

- *(b1) The maximum degree of $G[C_{X,\Delta}]$ is less than Δ, and*
- *(b2) The number of vertices not in $C_{X,\Delta} \cup C_{X,-\Delta}$ is at most $L2^L$.*

(c) For every pair of disjoint vertex subsets X_1 and X_2, each of size L, in G, if $L \ge L_0$ and $G[X_1 \cup X_2]$ is the complete bipartite graph with parts X_1 and X_2, then

- *(c1) Every vertex in $C_{X_1,\Delta} \cap C_{X_2,-\Delta}$ is adjacent to all but at most Δ vertices in $C_{X_1,-\Delta} \cap C_{X_2,\Delta}$, and*
- *(c2) The number of vertices in $C_{X_1,\Delta} \cap C_{X_2,\Delta}$ is less than Δ.*

Proof of (a1) Suppose on the contrary that G contains K_Δ as a subgraph. Let $\boldsymbol{v} \in \mathbb{R}^{V(G)}$ be the vector that assigns 1 to vertices in K_Δ and 0 otherwise. Then $\boldsymbol{v}^\intercal(\lambda I - A_G + \mu J)\boldsymbol{v}$ becomes

$$\lambda\Delta - \Delta(\Delta - 1) + \mu\Delta^2,$$

which would be negative if we had chosen $\Delta > (1+\lambda)/(1-\mu)$. □

Proof of (a2) Suppose on the contrary that G contains the complete $(p+1)$-partite graph $K_{\Delta,\dots,\Delta}$ as an induced subgraph. Again let $\boldsymbol{v} \in \mathbb{R}^{V(G)}$ be the vector that assigns 1 to the vertices in $K_{\Delta,\dots,\Delta}$ and 0 otherwise. Then $\boldsymbol{v}^\intercal(\lambda I - A_G + \mu J)\boldsymbol{v}$ becomes

$$\lambda(p+1)\Delta - p(p+1)\Delta^2 + \mu((p+1)\Delta)^2 = (p+1)\Delta\left(\lambda - (p - \mu(p+1))\Delta\right).$$

Because $p > 1/(1-\mu) - 1 = \mu/(1-\mu)$ or equivalently $p > \mu(p+1)$, the last factor above would be negative if we had chosen $\Delta > \lambda/(p - \mu(p+1))$. □

Proof of (b1) Suppose on the contrary that a vertex $u \in C_{X,\Delta}$ has Δ neighbors $v_1, \dots, v_\Delta \in C_{X,\Delta}$. Let $\boldsymbol{v} \in \mathbb{R}^{V(G)}$ be the vector that assigns L to u, $\lambda L/\Delta$ to $v_1, \dots, v_\Delta$, $-(\lambda+1)$ to the vertices in X, and 0 otherwise. Because $u, v_1, \dots, v_\Delta \in C_{X,\Delta}$, we have

$$\tfrac{1}{2}\boldsymbol{v}^\intercal A_G \boldsymbol{v} \geq \lambda L^2 - (\lambda+1)\Delta L - \lambda(\lambda+1)\Delta L = \lambda L^2 - (\lambda+1)^2\Delta L.$$

Using this bound and the fact that $\boldsymbol{v}^\intercal\mathbf{1} = 0$, we obtain that $\boldsymbol{v}^\intercal(\lambda I - A_G + \mu J)\boldsymbol{v}$ is at most

$$\begin{aligned}&\lambda(L^2 + \lambda^2 L^2/\Delta + (\lambda+1)^2 L) - 2(\lambda L^2 - (\lambda+1)^2\Delta L)\\ &\quad = -\lambda(1 - \lambda^2/\Delta)L^2 + O_{\lambda,\Delta}(L),\end{aligned}$$

which would be negative for sufficiently large L if we had chosen $\Delta > \lambda^2$. □

Proof of (b2) To show that $\left|V(G)\backslash(C_{X,\Delta} \cup C_{X,-\Delta})\right| \leq L2^L$, it suffices to prove $|C_X(A)| < L$ for every subset A of the independent set X such that $|A| > \Delta$ and $|X \setminus A| > \Delta$.

Write $a = |A|$, $b = |X\backslash A|$, and $c = |C_X(A)|$. For any $\alpha, \beta, \gamma \in \mathbb{R}$, we consider the vector $\boldsymbol{v} \in \mathbb{R}^{V(G)}$ that assigns α to the vertices in A, β to the vertices in $X \setminus A$, γ to the vertices in $C_X(A)$, and 0 otherwise, and we have

$$0 \leq \boldsymbol{v}^\intercal(\lambda I - A_G + \mu J)\boldsymbol{v} \leq \lambda(a\alpha^2 + b\beta^2 + c\gamma^2) - 2ac\alpha\gamma + \mu(a\alpha + b\beta + c\gamma)^2.$$

In particular, taking $\beta = -(a\alpha + c\gamma)/(b + \lambda/\mu)$, we obtain that for all $\alpha, \gamma \in \mathbb{R}$,

$$0 \leq \lambda(a\alpha^2 + c\gamma^2) - 2ac\alpha\gamma + \frac{\mu\lambda}{\mu b + \lambda}(a\alpha + c\gamma)^2.$$

For this quadratic form in α and γ to be positive semidefinite, its discriminant must be nonpositive:

$$\frac{(\mu b+\lambda(1-\mu))^2}{(\mu b+\lambda)^2}a^2c^2-\left(\lambda a+\frac{\mu\lambda a^2}{\mu b+\lambda}\right)\left(\lambda c+\frac{\mu\lambda c^2}{\mu b+\lambda}\right)\le 0,$$

which simplifies to

$$(\mu b+\lambda(1-\mu))^2ac\le\lambda^2(\mu a+\mu b+\lambda)(\mu b+\mu c+\lambda). \tag{1}$$

By the assumption that $a, b > \Delta$, if we had taken $\Delta \ge \max\{\lambda/\mu, 4\lambda^2, 2\}$, then $\lambda < \mu b$ and $\lambda^2 < b/4$, hence (1) would imply the following series of inequalities:

$$\begin{aligned}\mu^2ab^2c<(b/4)(\mu a+2\mu b)(2\mu b+\mu c)\implies abc<(a+b)(b+c)\\ \implies c<\frac{(a+b)b}{ab-a-b}\le a+b=L.\end{aligned}$$

□

Proof of (c1) Suppose on the contrary that a vertex $u \in C_{X_1,\Delta}\cap C_{X_2,-\Delta}$ is not adjacent to $v_1,\dots,v_\Delta \in C_{X_1,-\Delta}\cap C_{X_2,\Delta}$. Let $\boldsymbol{v}\in\mathbb{R}^{V(G)}$ be the vector that assigns L to v, $-\lambda L/\Delta$ to $v_1,\dots,v_\Delta$, -1 to the vertices in X_1, λ to the vertices in X_2, and 0 otherwise. Because $u\in C_{X_1,\Delta}\cap C_{X_2,-\Delta}$ and $v_1,\dots,v_\Delta\in C_{X_1,-\Delta}\cap C_{X_2,\Delta}$, we have

$$\tfrac{1}{2}\boldsymbol{v}^\intercal A_G\boldsymbol{v}\ge-\Delta L+\lambda(L-\Delta)L+(\lambda L/\Delta)\Delta(L-\Delta)-\lambda L^2=\lambda L^2-(2\lambda+1)\Delta L.$$

Using this bound and the fact that $\boldsymbol{v}^\intercal\mathbf{1}=0$, we obtain that $\boldsymbol{v}^\intercal(\lambda I-A_G+\mu J)\boldsymbol{v}$ is at most

$$\begin{aligned}&\lambda(L^2+\lambda^2L^2/\Delta+L+\lambda^2L)-2(\lambda L^2-(2\lambda+1)\Delta L)\\ &\quad=-\lambda(1-\lambda^2/\Delta)L^2+O_{\lambda,\Delta}(L),\end{aligned}$$

which would be negative for sufficiently large L if we had chosen $\Delta>\lambda^2$. □

Proof of (c2) Suppose on the contrary that $C_{X_1,\Delta}\cap C_{X_2,\Delta}$ contains $v_1,\dots,v_\Delta$. Let $\boldsymbol{v}\in\mathbb{R}^{V(G)}$ be the vector that assigns $2L/\Delta$ to $v_1,\dots,v_\Delta$, -1 to the vertices in $X_1\cup X_2$, and 0 otherwise. Because $v_1,\dots,v_\Delta\in C_{X_1,\Delta}\cap C_{X_2,\Delta}$, we have

$$\tfrac{1}{2}\boldsymbol{v}^\intercal A_G v\ge-(2L/\Delta)2\Delta^2+L^2=-4\Delta L+L^2.$$

Using this bound and the fact that $\boldsymbol{v}^\intercal\mathbf{1}=0$, we obtain that $\boldsymbol{v}^\intercal(\lambda I-A_G+\mu J)\boldsymbol{v}$ is at most

$$\lambda(4L^2/\Delta+2L)-2(-4\Delta L+L^2)=-2(1-2\lambda/\Delta)L^2+O_{\lambda,\Delta}(L),$$

which would be negative for sufficiently large L if we had chosen $\Delta>2\lambda$. □

Proof of Theorem 1.7 Let $\lambda = (1-\alpha)/(\alpha-\beta)$ and $\mu = \alpha/(\alpha-\beta)$ (and so $p = \lfloor -\alpha/\beta + 1\rfloor = \lfloor 1/(1-\mu)\rfloor$). As in Lemma 2.1, the associated graph G of the spherical $\{\alpha, \beta\}$-set satisfies $\lambda I - A_G + \mu J \succeq 0$.

Choose Δ and L_0 as in Lemma 3.2. We shall prove that G, after removing at most $pL2^L + \binom{p}{2}\Delta + R(\Delta, L2^{pL})$ vertices, is a $p\Delta$-modification of a complete p-partite graph, where $L = L_0 + (p+2)\Delta$ and $R(\cdot,\cdot)$ is the Ramsey number.

We may assume that $|G| \geq R(\Delta, L)$ because otherwise G is vacuously a $p\Delta$-modification of a complete p-partite graph after removing all its vertices. By Lemma 3.2(a1) and Ramsey's theorem, there exists an independent set of size L in G. Choose the maximum $t \leq p$ such that the complete t-partite graph $K_{L-t\Delta,\ldots,L-t\Delta}$ is an induced subgraph of G (note that $t \geq 1$ since there is an independent set of size L). Let $X_1, \ldots, X_t \subset V(G)$ be the parts of this t-partite graph.

Define for every $i \in \{1, \ldots, t\}$ the vertex subset

$$V_i = C_{X_i,\Delta} \cap \bigcap_{j \neq i} C_{X_j,-\Delta}.$$

By (b1) and (c1) in Lemma 3.2, we see that the $G[V_1 \cup \cdots \cup V_t]$ is a $t\Delta$-modification of the complete t-partite graph with parts $V_1, \ldots, V_t$.

We bound $U := V(G) \setminus (V_1 \cup \cdots \cup V_t)$ as follows. Set

$$U_i = V(G) \setminus (C_{X_i,\Delta} \cup C_{X_i,-\Delta}), \quad U_{ij}^- = C_{X_i,\Delta} \cap C_{X_j,\Delta}, \quad U^+ = \bigcap_i C_{X_i,-\Delta}.$$

Note that $U = (\bigcup_i U_i) \cup (\bigcup_{i<j} U_{ij}^-) \cup U^+$. It is enough to bound the cardinalities of U_i, U_{ij}, U^+. Lemma 3.2(b2) says that $|U_i| \leq L2^L$ for each i. Lemma 3.2(c2) says that $\left|U_{ij}^-\right| \leq \Delta$ for $i < j$.

Finally, we claim that U^+ does not contain a subset of size $L2^{tL}$ that is independent in G. Indeed, suppose on the contrary that U^+ contains an independent set of size $L2^{tL}$. Since every vertex in U^+ has at least $L-(t+1)\Delta$ neighbors in X_i for each i, by the pigeonhole principle, there exist $X_1' \subseteq X_1, \ldots, X_t' \subseteq X_t$ and $U' \subseteq U^+$, each of size $L-(t+1)\Delta$, such that $G[X_1' \cup \cdots \cup X_t' \cup U']$ is a complete $(t+1)$-partite graph with parts $X_1', \ldots, X_t'$ and U', which contradicts our choice of t or Lemma 3.2(a2) in case $t = p$. This finishes the proof of the claim. In view of Lemma 3.2(a1) and Ramsey's theorem, we obtain $\left|U^+\right| < R(\Delta, L2^{tL})$. In total, $|U| \leq tL2^L + \binom{t}{2}\Delta + R(\Delta, L2^{tL})$. □

4 Graph Eigenvalue Multiplicity Argument

We estimate the eigenvalue multiplicity of a signed graph with bounded maximum degree by that of a (not necessarily connected) graph. Recall Definition 1.2 of the spectral radius order $k(\lambda)$.

Lemma 4.1 *For every $\lambda > 0$, $\Delta \in \mathbb{N}$, and $j \in \mathbb{N}$, if G is an n-vertex graph with maximum degree at most Δ and $\lambda_j(G) \le \lambda$, then*

$$\operatorname{mult}(\lambda, G) \le \begin{cases} n/k(\lambda) + O_{\Delta,j,\lambda}(1) & \text{if } k(\lambda) < \infty, \\ O_{\Delta,j}(n/\log\log n) & \text{otherwise.} \end{cases}$$

Proof Let $G_1, \ldots, G_t$ be the connected components of G numbered such that $\lambda_1(G_1), \ldots, \lambda_1(G_s) > \lambda$ and $\lambda_1(G_{s+1}), \ldots, \lambda_1(G_t) \le \lambda$. Because $\lambda_j(G) \le \lambda$, we know that $s < j$. Set $n_i = |G_i|$ and $n = \sum n_i = |G|$.

For each $i \le s$, since G_i is a connected graph with maximum degree at most Δ and $\lambda_j(G_i) \le \lambda$, Theorem 1.5 gives a constant $C = C(\Delta, j)$ such that

$$\operatorname{mult}(\lambda, G_i) \le \frac{Cn_i}{\log\log n_i}. \tag{2}$$

We break the rest of the proof into two cases.

Case $k(\lambda) < \infty$. Set $N_0 = \exp(\exp(Ck(\lambda)))$. For $i \le s$, when $n_i \ge N_0$, we can relax (2) to $\operatorname{mult}(\lambda, G_i) \le n_i/k(\lambda)$; when $n_i < N_0$, clearly $\operatorname{mult}(\lambda, G_i) \le n_i < N_0$. To sum up, for $i \le s$, we always have

$$\operatorname{mult}(\lambda, G_i) \le \frac{n_i}{k(\lambda)} + N_0. \tag{3}$$

For each $i > s$, when $\lambda_1(G_i) = \lambda$, because G_i is connected, we know that $n_i \ge k(\lambda)$, and so by the Perron–Frobenius theorem, we obtain

$$\operatorname{mult}(\lambda, G_i) \le 1 \le \frac{n_i}{k(\lambda)}; \tag{4}$$

when $\lambda_1(G_i) < \lambda$, clearly (4) holds trivially. We combine (3) and (4) to obtain

$$\operatorname{mult}(\lambda, G) = \sum_{i=1}^{t} \operatorname{mult}(\lambda, G_i) \le \sum_{i=1}^{t} \frac{n_i}{k(\lambda)} + sN_0 \le \frac{n}{k(\lambda)} + O_{\Delta,j,\lambda}(1).$$

Case $k(\lambda) = \infty$. For $i > s$, because $\lambda_1(G_i) \le \lambda$ and $k(\lambda) = \infty$, it must be the case that $\lambda_1(G_i) < \lambda$, and so $\operatorname{mult}(\lambda, G_i) = 0$. Therefore (2) gives

$$\operatorname{mult}(\lambda, G) = \sum_{i=1}^{s} \operatorname{mult}(\lambda, G_i) \le j \cdot \max_{1 \le i \le j} \frac{Cn_i}{\log\log n_i} = O_{\Delta,j}\left(\frac{n}{\log\log n}\right). \qquad \square$$

Next we prove Theorem 1.13, which states that

$$N_{\alpha,\beta}(d) \le \begin{cases} \dfrac{qk(\lambda)d}{k(\lambda)-1} + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ qd + o(d) & \text{otherwise,} \end{cases}$$

where $\lambda = (1-\alpha)/(\alpha-\beta)$ and $p = \lfloor -\alpha/\beta \rfloor + 1$ and $q = \max\{1, p/2\}$.

Proof of Theorem 1.13 In view of Lemma 2.1, consider a graph $\widetilde{G}$ on $N_{\alpha,\beta}(d)$ vertices satisfying

$$\lambda I - A_{\widetilde{G}} + \mu J \succeq 0 \quad \text{and} \quad \operatorname{rank}\bigl(\lambda I - A_{\widetilde{G}} + \mu J\bigr) \le d,$$

where $\lambda = (1-\alpha)/(\alpha-\beta)$ and $\mu = \alpha/(\alpha-\beta)$. By Theorem 1.7 we obtain a constant $\Delta = \Delta(\alpha, \beta)$ such that the graph, denoted G, obtained from $\widetilde{G}$ by removing at most Δ vertices is a Δ-modification of a complete p-partite graph, denoted K, where $p = \lfloor 1/(1-\mu) \rfloor$. Define the signed graph $G^{\pm}$ by $A_{G^{\pm}} = A_G - A_K$. Notice that the maximum degree of $G^{\pm}$ is at most Δ, and $\chi(G^{\pm}) \le p$.

Now the signed adjacency matrix of $G^{\pm}$ satisfies

$$\lambda I - A_{G^{\pm}} + \mu J - A_K \succeq 0 \quad \text{and} \quad \operatorname{rank}(\lambda I - A_{G^{\pm}} + \mu J - A_K) \le d.$$

Note that $\operatorname{rank}(\mu J - A_K) \le p$. From the first condition above, we deduce using the Courant–Fischer theorem that $\lambda_{p+1}(\lambda I - A_{G^{\pm}}) \ge 0$ or equivalently $\lambda_{p+1}(G^{\pm}) \le \lambda$. From the second condition above, we deduce using subadditivity of matrix ranks that $\operatorname{rank}(\lambda I - A_{G^{\pm}}) \le d + p$ or equivalently

$$\operatorname{mult}(\lambda, G^{\pm}) \ge \left|G^{\pm}\right| - (d+p). \tag{5}$$

We break the rest of the proof into two cases.

Case $p = 1$. The signed graph $G^{\pm}$ consists of positive edges only. Lemma 4.1 provides the upper bound

$$\operatorname{mult}(\lambda, G^{\pm}) \le \begin{cases} \left|G^{\pm}\right|/k(\lambda) + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ o(\left|G^{\pm}\right|) & \text{otherwise.} \end{cases}$$

Combining with (5), we get

$$\left|G^{\pm}\right| - (d+p) \le \begin{cases} \left|G^{\pm}\right|/k(\lambda) + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ o(\left|G^{\pm}\right|) & \text{otherwise,} \end{cases}$$

which implies

$$\left|G^{\pm}\right| \le \begin{cases} \dfrac{k(\lambda)d}{k(\lambda)-1} + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ d + o(d) & \text{otherwise.} \end{cases}$$

The desired upper bound on $N_{\alpha,\beta}(d)$ follows immediately in view of $\left|G^{\pm}\right| \ge N_{\alpha,\beta}(d) - \Delta$.

Case $p \ge 2$. Let V_1 and V_2 be the largest parts of the complete p-partite graph K. Let $G^{\pm}_{12}$ be the signed subgraph of $G^{\pm}$ induced on $V_1 \cup V_2$, and let G_{12} be the underlying graph of $G^{\pm}_{12}$. Notice that $|G_{12}| = |V_1| + |V_2| \geq 2\left|G^{\pm}\right|/p$, and the maximum degree of G_{12} is at most Δ, and $\chi(G^{\pm}_{12}) \leq 2$. Since $\chi(G^{\pm}_{12}) \leq 2$, the signed graph $G^{\pm}_{12}$ is isospectral to its underlying graph G_{12}. It follows from Lemma 4.1 that

$$\operatorname{mult}(\lambda, G^{\pm}_{12}) = \operatorname{mult}(\lambda, G_{12}) \leq \begin{cases} |G_{12}|/k(\lambda) + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ o(|G_{12}|) & \text{otherwise.} \end{cases}$$

By the Cauchy interlacing theorem, we have

$$\operatorname{mult}(\lambda, G^{\pm}) - (\left|G^{\pm}\right| - |G_{12}|) \leq \operatorname{mult}(\lambda, G^{\pm}_{12}).$$

Combining (5) and the above two inequalities, we get

$$\begin{aligned} |G_{12}| - (d+p) &\overset{(5)}{\leq} \operatorname{mult}(\lambda, G^{\pm}) - (\left|G^{\pm}\right| - |G_{12}|) \\ &\leq \begin{cases} |G_{12}|/k(\lambda) + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ o(|G_{12}|) & \text{otherwise,} \end{cases} \end{aligned}$$

which implies

$$|G_{12}| \leq \begin{cases} \dfrac{k(\lambda)d}{k(\lambda)-1} + O_{\alpha,\beta}(1) & \text{if } k(\lambda) < \infty, \\ d + o(d) & \text{otherwise.} \end{cases}$$

The desired upper bound on $N_{\alpha,\beta}(d)$ follows immediately in view of the inequalities $|G_{12}| \geq 2\left|G^{\pm}\right|/p$ and $\left|G^{\pm}\right| \geq N_{\alpha,\beta}(d) - \Delta$. □

As a corollary, we obtain the following general lower bound on $k_p(\lambda)$.

Corollary 4.2 *For all* $\lambda > 0$ *and* $p \geq 2$*,*

$$k_p(\lambda) \geq \frac{pk(\lambda)}{pk(\lambda) - 2\lambda}.$$

Proof Comparing Proposition 2.2 and Theorem 1.13, we get

$$\frac{k_p(\lambda)d}{k_p(\lambda)-1} - o(d) \leq \frac{pk(\lambda)d}{2(k(\lambda)-1)} + O_{p,\lambda}(1),$$

which implies the desired lower bound. (It is also not hard to prove Corollary 4.2 directly, but we do not do so here.) □

Remark For general λ, we do not know any algorithm for computing $k(\lambda)$ (or even deciding whether $k(\lambda) < \infty$), though deciding whether $k(\lambda) < k$ for each integer k is a finite problem as can be done by a brute-force search over all graphs up to a fixed size.

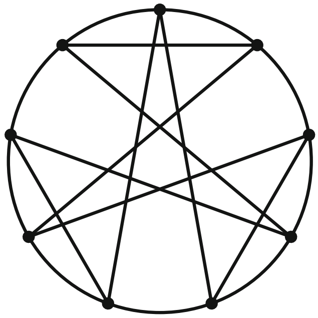


Fig. 2 The Paley graph of order 9

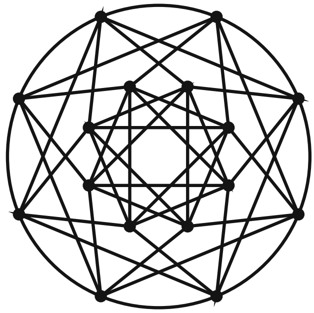


Fig. 3 The Shrikhande graph

When $\lambda \in \mathbb{N}$, we have $k(\lambda) = \lambda + 1$ because the complete graph $K_{\lambda+1}$ is the graph on fewest vertices with spectral radius λ. In contrast, even for $\lambda \in \mathbb{N}$, computing the exact values of $k_p(\lambda)$ seems to be very difficult for $p \geq 3$. For $\lambda = 2$, Corollary 4.2 implies that $k_3(2) \geq 9/5$ and $k_4(2) \geq 3/2$. Note that both the Paley graph of order 9 in Fig. 2 and the Shrikhande graph in Fig. 3 are strongly regular graphs with -2 as their smallest eigenvalue with multiplicity 4 and 9 respectively. Moreover their chromatic numbers are 3 and 4 respectively. The all-negative signed graphs of these two strongly regular graphs would yield $k_3(2) \leq 9/4$ and $k_4(2) \leq 16/9$. We leave the determination of $k_p(2)$ for $p \geq 3$ as an open problem.

Theorems 1.12(a) and 1.12(b) follow easily from Theorem 1.13 and Proposition 2.2.

Proof of Theorem 1.12(a) Because $p \leq 2$, we have $q = \max\{1, p/2\} = 1$ and $k_p(\lambda) = k(\lambda)$. Moreover, if $k(\lambda) < \infty$ then $k(\lambda)$ can be achieved for $k_p(\lambda)$ by the smallest graph whose spectral radius is exactly λ. Thus Theorem 1.13 and Proposition 2.2 give matching bounds on $N_{\alpha,\beta}(d)$. □

Proof of Theorem 1.12(b) Because $\lambda = 1$ and $p \geq 2$, we have $k(\lambda) = 2$ and $q = \max(1, p/2) = p/2$. Thus Theorem 1.13 gives

$$N_{\alpha,\beta}(d) \leq pd + O_{\alpha,\beta}(1). \tag{6}$$

Corollary 4.2 implies that $k_p(1) \geq p/(p-1)$. To see that $p/(p-1)$ can be achieved for $k_p(1)$, consider the all-negative complete signed graph $K_p^{\pm}$ on p vertices. Clearly

$\chi(K_p^{\pm}) = p$. Since the smallest eigenvalue of the complete unsigned graph K_p is -1 with multiplicity $p-1$, the largest eigenvalue of $K_p^{\pm}$ is 1 with multiplicity $p-1$. Now Proposition 2.2 provides a lower bound that matches (6) up to an additive constant. □

5 Forbidden Induced Subgraphs

The next lemma enables us to forbid finitely many induced subgraphs in the signed graph that arises from Theorem 1.7. Here an *induced subgraph* of a signed graph keeps the original edge signs.

Lemma 5.1 *Fix $\lambda > 0$, $\mu \in (0, 1)$, $p \in \mathbb{N}$, and $\Delta \in \mathbb{N}$. For every signed graph $H^{\pm}$ with $\lambda_1(H^{\pm}) > \lambda$, there exists $n_0 \in \mathbb{N}$ such that for every $t \le p$ and every graph G that is a Δ-modification of a complete t-partite graph K, if $\lambda I - A_G + \mu J \succeq 0$, and the size of each part of K is at least n_0, then $H^{\pm}$ cannot be an induced subgraph of the signed graph $G^{\pm}$ defined by $A_{G^{\pm}} = A_G - A_K$.*

Proof Suppose that G is a Δ-modification of a complete t-partite graph K with parts $\widetilde{V}_1, \dots, \widetilde{V}_t$, and suppose that the size of each part of K is at least n_0. Assume for the sake of contradiction that $H^{\pm}$ with $\lambda_1(H^{\pm}) > \lambda$ is an induced subgraph of $G^{\pm}$. Take $n_0 = (\left|H^{\pm}\right| + pm)\Delta$, where $m = \lfloor \lambda \left|H^{\pm}\right| / (\lambda_1(H^{\pm}) - \lambda) \rfloor + 1$. We can greedily find $V_1 \subseteq \widetilde{V}_1, \dots, V_t \subseteq \widetilde{V}_t$ such that

(1) each V_i is disjoint from $V(H^{\pm})$ and has size m,
(2) G induces a complete t-partite graph with parts $V_1, \dots, V_t$,
(3) for every vertex v of $H^{\pm}$, if $v \in \widetilde{V}_i$, then, in G, the vertex v is adjacent to every vertex in V_j for $j \ne i$, and is not adjacent to any vertex in V_i.

Let $\boldsymbol{x} \in \mathbb{R}^{V(H^{\pm})}$ be a top eigenvector of $H^{\pm}$, and set

$$s_i = \sum_{u \in V(H^{\pm}) \cap \widetilde{V}_i} \boldsymbol{x}_u.$$

Note that $s_i^2 \le \left|V(H^{\pm}) \cap \widetilde{V}_i\right| \boldsymbol{x}^{\intercal}\boldsymbol{x}$ for each i, which implies that

$$\sum_i s_i^2 \le \left|H^{\pm}\right| \boldsymbol{x}^{\intercal}\boldsymbol{x}. \tag{7}$$

Consider the vector $\boldsymbol{v} \in \mathbb{R}^{V(G)}$ extending $\boldsymbol{x}$ that in addition assigns $-s_i/m$ to each vertex in V_i for $i \in \{1, \dots, t\}$. Since $\boldsymbol{v}$ is chosen so that $\sum_{u \in \widetilde{V}_i} \boldsymbol{v}_u = 0$ for each $i \in \{1, \dots, t\}$, we have $J\boldsymbol{v} = 0$ and $A_K \boldsymbol{v} = 0$. Now we can simplify the quadratic form as follows:

$$\boldsymbol{v}^{\intercal}(\lambda I - A_G + \mu J)\boldsymbol{v} = \boldsymbol{v}^{\intercal}(\lambda I - A_{G^{\pm}} - A_K + \mu J)\boldsymbol{v} = \boldsymbol{v}^{\intercal}(\lambda I - A_{G^{\pm}})\boldsymbol{v}.$$

Next, since no vertex in $H^{\pm}$ is adjacent to $V_1 \cup \cdots \cup V_t$ in $G^{\pm}$, we have

$$\begin{aligned}\boldsymbol{v}^{\intercal}(\lambda I - A_{G^{\pm}})\boldsymbol{v} &= \boldsymbol{x}^{\intercal}(\lambda I - A_{H^{\pm}})\boldsymbol{x} + \lambda \sum_i m(s_i/m)^2 \\ &= \left(\lambda - \lambda_1(H^{\pm})\right)\boldsymbol{x}^{\intercal}\boldsymbol{x} + \lambda \sum_i s_i^2/m \\ &\leq \left(\lambda - \lambda_1(H^{\pm}) + \lambda \left|H^{\pm}\right|/m\right)\boldsymbol{x}^{\intercal}\boldsymbol{x}, \qquad by \quad (7)\end{aligned}$$

which is negative because $m > \lambda \left|H^{\pm}\right|/(\lambda_1(H^{\pm}) - \lambda)$. This contradicts $\lambda I - A_G + \mu J \succeq 0$. □

Lemma 5.1 leads us to bound eigenvalue multiplicities in a restricted class of signed graphs obtained by forbidding certain induced subgraphs.

Definition 5.2 Given a family $\mathcal{H}$ of signed graphs, let $M_{p,\mathcal{H}}(\lambda, N)$ be the maximum possible value of $\operatorname{mult}(\lambda, G^{\pm})$ over all signed graphs $G^{\pm}$ on at most N vertices that do not contain any member of $\mathcal{H}$ as an induced subgraph and satisfy $\chi(G^{\pm}) \leq p$ and $\lambda_{p+1}(G^{\pm}) \leq \lambda$.

In our application, we will only be allowed to forbid a *finite* $\mathcal{H}$ such that $\lambda_1(H^{\pm}) > \lambda$ for all $H^{\pm} \in \mathcal{H}$.

Remark We could choose $\mathcal{H}$ properly so that every signed graph $G^{\pm}$ considered in Definition 5.2 of $M_{p,\mathcal{H}}(\lambda, N)$ has its maximum degree bounded by a constant depending only on p and λ. In fact, set $D = \lfloor \lambda^2 \rfloor$, and suppose that $\mathcal{H}$ includes all the signed graphs $H^{\pm}$ on $D+2$ vertices with $\chi(H^{\pm}) \leq 2$ such that the underlying graph of $H^{\pm}$ contains the star $K_{1,D+1}$. One can then show that for every graph $G^{\pm}$ that does not contain any member of $\mathcal{H}$ as an induced subgraph, the maximum degree of $G^{\pm}$ is at most $\chi(G^{\pm})D$.

The next statement relates the maximum size of a spherical two-distance set with the above eigenvalue multiplicity quantity.

Theorem 5.3 *Fix $-1 \leq \beta < 0 \leq \alpha < 1$. Set $\lambda = (1-\alpha)/(\alpha - \beta)$ and $p = \lfloor -\alpha/\beta \rfloor + 1$. Let $\mathcal{H}$ be a finite family of signed graphs with $\lambda_1(H^{\pm}) > \lambda$ for each $H^{\pm} \in \mathcal{H}$. Then*

$$N_{\alpha,\beta}(d) \leq d + M_{p,\mathcal{H}}(\lambda, N_{\alpha,\beta}(d)) + O_{\alpha,\beta,\mathcal{H}}(1).$$

Proof In view of Lemma 2.1, consider a graph $\widetilde{G}$ on $N_{\alpha,\beta}(d)$ vertices satisfying

$$\lambda I - A_{\widetilde{G}} + \mu J \succeq 0 \quad \text{and} \quad \operatorname{rank}(\lambda I - A_{\widetilde{G}} + \mu J) \leq d,$$

where $\lambda = (1-\alpha)/(\alpha-\beta)$ and $\mu = \alpha/(\alpha-\beta)$. By Lemma 3.2 we obtain a constant $\Delta = \Delta(\alpha, \beta)$ such that $\widetilde{G}$, after removing at most Δ vertices, is a Δ-modification of a complete p-partite graph, where $p = \lfloor 1/(1-\mu) \rfloor$.

Let $n_0 = n_0(\alpha, \beta, \mathcal{H})$ be the maximum n_0 given by Lemma 5.1 when it is applied to each member of $\mathcal{H}$ respectively with the parameters λ, μ, p, and Δ. After removing at most Δ vertices from $\widetilde{G}$, we can further remove at most pn_0 vertices from $\widetilde{G}$ to obtain a graph, denoted G, that is a Δ-modification of a t-partite graph, denoted K, with each part of size at least n_0, for some $t \le p$. Define the signed graph $G^{\pm}$ by $A_{G^{\pm}} = A_G - A_K$. Since $\lambda I - A_G + \mu J \succeq 0$, by our choice of n_0, we know that the signed graph $G^{\pm}$ does not contain any member of $\mathcal{H}$ as an induced subgraph. Notice that $\chi(G^{\pm}) \le t \le p$.

Now the signed adjacency matrix of $G^{\pm}$ satisfies

$$\lambda I - A_{G^{\pm}} + \mu J - A_K \succeq 0, \tag{8a}$$

$$\operatorname{rank}(\lambda I - A_{G^{\pm}} + \mu J - A_K) \le d. \tag{8b}$$

Note that $\operatorname{rank}(\mu J - A_K) \le t \le p$. From (8a) we deduce using the Courant–Fischer theorem that $\lambda_{p+1}(\lambda I - A_{G^{\pm}}) \ge 0$ or equivalently $\lambda_{p+1}(G^{\pm}) \le \lambda$. Recall that $G^{\pm}$ has at most $N_{\alpha,\beta}(d)$ vertices, $G^{\pm}$ does not contain any member of $\mathcal{H}$ as an induced subgraph, and $\chi(G^{\pm}) \le p$. According to Definition 5.2,

$$\operatorname{mult}(\lambda, G^{\pm}) \le M_{p,\mathcal{H}}(\lambda, N_{\alpha,\beta}(d)).$$

From (8b) we deduce using subadditivity of matrix ranks that $\operatorname{rank}(\lambda I - A_{G^{\pm}}) \le d + p$ or equivalently

$$\operatorname{mult}(\lambda, G^{\pm}) \ge \left|G^{\pm}\right| - (d + p).$$

Combining with $\left|G^{\pm}\right| \ge N_{\alpha,\beta}(d) - \Delta - pn_0$, we get

$$\begin{aligned} N_{\alpha,\beta}(d) &\le \left|G^{\pm}\right| + \Delta + pn_0 \\ &\le d + \operatorname{mult}(\lambda, G^{\pm}) + \Delta + p(n_0 + 1) \\ &\le d + M_{p,\mathcal{H}}(\lambda, N_{\alpha,\beta}(d)) + O_{\alpha,\beta,\mathcal{H}}(1). \end{aligned}$$

□

For each value of λ and p, if we could prove the following upper bound on the eigenvalue multiplicity, then it would imply Conjecture 1.11 via Theorem 5.3.

Conjecture 5.4 *For every $\lambda > 0$ and $p \in \mathbb{N}$, there exists a finite family $\mathcal{H}$ of signed graphs with $\lambda_1(H^{\pm}) > \lambda$ for each $H^{\pm} \in \mathcal{H}$ such that*

$$M_{p,\mathcal{H}}(\lambda, N) \le \begin{cases} N/k_p(\lambda) + o(N) & \text{if } k_p(\lambda) < \infty, \\ o(N) & \text{otherwise.} \end{cases}$$

We include the short deduction below that for each $\lambda > 0$ and $p \in \mathbb{N}$, Conjecture 5.4 implies Conjecture 1.11. Though, for deducing Theorem 1.12(c) in the next section, we will prove each bound directly without resorting to Conjecture 5.4, in order to give a slightly better error term of $O_{\alpha,\beta}(1)$ instead of $o(d)$.

Proof that Conjecture 5.4 implies Conjecture 1.11 for each $\lambda > 0$ and $p \in \mathbb{N}$ Choose $\mathcal{H}$ as in Conjecture 5.4. In the case when $k_p(\lambda) < \infty$, by Theorem 5.3, we have

$$N_{\alpha,\beta}(d) \le d + M_{p,\mathcal{H}}(\lambda, N_{\alpha,\beta}(d)) + O_{\alpha,\beta}(1) \le d + \left(\frac{1}{k_p(\lambda)} + o(1)\right) N_{\alpha,\beta}(d).$$

Therefore

$$N_{\alpha,\beta}(d) \le \left(\frac{k_p(\lambda)}{k_p(\lambda) - 1} + o(1)\right) d,$$

which matches the lower bound in Proposition 2.2. The case of $k_p(\lambda) = \infty$ is similar. □

6 Third Moment Argument

For $\lambda = \sqrt{3}$ and $p = 3$, we give a tight upper bound (verifying Conjecture 5.4) on $\text{mult}(\lambda, G^\pm)$ for those signed graphs $G^\pm$ in Theorem 5.3, which implies a tight upper bound on the corresponding $N_{\alpha,\beta}(d)$.

Theorem 6.1 *There exists a finite family $\mathcal{H}$ of signed graphs with $\lambda_1(H^\pm) > \sqrt{3}$ for each $H^\pm \in \mathcal{H}$ such that*

$$M_{3,\mathcal{H}}(\sqrt{3}, N) \le 3N/7.$$

Proof Let $\mathcal{H}$ be the family of all the signed graphs $H^\pm$ on at most 5 vertices with $\lambda_1(H^\pm) > \sqrt{3}$. For the sake of contradiction, assume that $G^\pm$ is a signed graph with the minimum number of vertices such that $\chi(G^\pm) \le 3$, no member of $\mathcal{H}$ is an induced subgraph of $G^\pm$, and $\text{mult}(\sqrt{3}, G^\pm) > 3\left|G^\pm\right|/7$. By our choice of $\mathcal{H}$, every subgraph of $G^\pm$ induced by at most 5 vertices has largest eigenvalue at most $\sqrt{3}$. Note that $G^\pm$ is connected by its minimality. Let $V(G^\pm) = V_1 \sqcup V_2 \sqcup V_3$ be a valid 3-coloring of $G^\pm$ allowing some V_i's to be empty, and let G be the underlying graph of $G^\pm$. The next four claims reveal the local structure of $G^\pm$.

Claim 1 *The edges of every triangle in G are all negative in $G^\pm$.*

Proof of Claim 1 Since $\chi(G^\pm)$ is finite, every signed triangle in $G^\pm$, other than the all negative one, contains 0 or 2 negative edges. In either case, the chromatic number of the signed triangle is 2, hence its largest eigenvalue equals $\lambda_1(K_3) = 2$. However, every induced triangle of $G^\pm$ has largest eigenvalue at most $\sqrt{3}$. □

Claim 2 *If G induces a star on $\{v_0, v_1, v_2, v_3\}$ centered at v_0, then v_1, v_2, v_3 are the only neighbors of v_0 in G, and moreover for every $w \ne v_0$ that is adjacent to at least one of v_1, v_2, v_3, exactly two of v_1, v_2, v_3 are adjacent to w in G.*

Proof of Claim 2 Let $w \in V(G) \setminus \{v_0, v_1, v_2, v_3\}$ be a vertex that is adjacent to at least one of v_0, v_1, v_2, v_3, and consider the vector $\boldsymbol{v} \in \mathbb{R}^W$, where $W = \{v_0, v_1, v_2, v_3, w\}$, that assigns $\sqrt{3}$ to v_0, $\sigma(v_0 v_i)$ to v_i for $i \in \{1, 2, 3\}$, ε to w, where $\sigma : E(G) \to \{\pm 1\}$ is the signing of $G^{\pm}$ and $\varepsilon \in \mathbb{R}$. According to our choice of $\boldsymbol{v}$, we have

$$\boldsymbol{v}^{\mathsf{T}} A_{G^{\pm}[W]} \boldsymbol{v} = 6\sqrt{3} + 2\varepsilon \sum_{v_i w \in E(G)} \sigma(v_i w) \boldsymbol{v}_{v_i}.$$

By the Courant–Fischer theorem, we also have

$$\boldsymbol{v}^{\mathsf{T}} A_{G^{\pm}[W]} \boldsymbol{v} \le \lambda_1(G^{\pm}[W]) \boldsymbol{v}^{\mathsf{T}} \boldsymbol{v} \le \sqrt{3}(6 + \varepsilon^2).$$

For the last inequality to hold for all $\varepsilon \in \mathbb{R}$, we must have

$$\sum_{v_i w \in E(G)} \sigma(v_i w) \boldsymbol{v}_{v_i} = 0,$$

which implies that $v_0 w \notin E(G)$, and exactly two of v_1, v_2, v_3 are adjacent to w in G. □

Claim 3 *The maximum degree of G is at most* 4.

Proof of Claim 3 Suppose on the contrary that v_0 is adjacent to at least 5 vertices in G. Without loss of generality we may assume that $v_0 \in V_1$, and by the pigeonhole principle that 3 neighbors, say v_1, v_2, v_3, of v_0 are in $V_1 \cup V_2$. As $\chi(H^{\pm}) \le 2$, where $H^{\pm} := G^{\pm}[\{v_0, v_1, v_2, v_3\}]$, by Claim 1, $H^{\pm}$ contains no triangles. Thus G induces a star on $\{v_0, v_1, v_2, v_3\}$ centered at v_0, and so by Claim 2, v_0 has no neighbors other than v_1, v_2, v_3 in G, which leads to a contradiction. □

Claim 4 *The underlying graph G contains an induced star* $K_{1,3}$.

Proof of Claim 4 Suppose on the contrary that G does not contain any induced $K_{1,3}$. For every $v \in V(G)$, the subgraph of G induced by the neighbors of v contains no independent set of size 3, in particular, this induced subgraph contains at most 2 connected components, hence it contains at least $d_v - 2$ edges, where d_v is the degree of v in G. In other words, every $v \in V(G)$ is contained in at least $d_v - 2$ triangles.

Recall from Claim 2 that every triangle in G has all its edges negatively signed. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $G^{\pm}$, where $n = \left|G^{\pm}\right|$, and let t be the total number of triangles in G. Thus we have

$$-\sum_i \lambda_i^3 = -\operatorname{tr}(A_{G^{\pm}}^3) = 6t \ge 2\sum_v (d_v - 2).$$

Note that

$$\sum_i \lambda_i^2 = \operatorname{tr}(A_{G^{\pm}}^2) = \sum_v d_v \quad \text{and} \quad \sum_i \lambda_i = \operatorname{tr}(A_{G^{\pm}}) = 0.$$

Thus we have

$$\sum_i(\lambda_i^3+2\lambda_i^2-7\lambda_i)\le -2\sum_v(d_v-2)+2\sum_v d_v=4n.$$

Since the characteristic polynomial of $A_{G^\pm}$ is a polynomial with integer coefficients, we obtain $\text{mult}(-\sqrt{3},G^\pm)=\text{mult}(\sqrt{3},G^\pm)$, which is more than $3n/7$. For other eigenvalues λ_i, by Claim 3, we know that $\lambda_i\ge -4$, and so

$$\lambda_i^3+2\lambda_i^2-7\lambda_i=(\lambda_i-1)^2(\lambda_i+4)-4\ge -4.$$

Therefore

$$\sum_i(\lambda_i^3+2\lambda_i^2-7\lambda_i)>\frac{3n}{7}\cdot 2\cdot 2(\sqrt{3})^2+\frac{n}{7}\cdot(-4)=\frac{32n}{7}>4n,$$

which is a contradiction. □

The following claim imposes restriction on $G^\pm$ with small number of vertices.

Claim 5 *The number n of vertices in G is either* 6 *or at least* 8. *Moreover, if $n\in\{6,8\}$ then G is a* 3*-regular graph, and the signed adjacency matrix of $G^\pm$ satisfies $A_{G^\pm}^2=3I$.*

Proof Claim 4 shows that $n\ge 4$, and moreover when $n=4$, G is precisely $K_{1,3}$, in which case $\text{mult}(\sqrt{3},G^\pm)=\text{mult}(\sqrt{3},G)=1\le 3n/7$. Thus $n\ge 5$. Because $\text{mult}(-\sqrt{3},G^\pm)=\text{mult}(\sqrt{3},G^\pm)>3n/7$, we obtain

$$n\ge \text{mult}(\sqrt{3},G^\pm)+\text{mult}(-\sqrt{3},G^\pm)\ge 2(\lfloor 3n/7\rfloor+1), \tag{9}$$

which rules out $n=5$ and $n=7$. Therefore $n=6$ or $n\ge 8$. Suppose that $n\in\{6,8\}$. Note that equality must hold for (9). Thus $\text{mult}(-\sqrt{3},G^\pm)=\text{mult}(\sqrt{3},G^\pm)=n/2$, which implies that $\text{mult}(3,A_{G^\pm}^2)=n$. Hence $A_{G^\pm}^2=3I$, which in particular implies that G is a 3-regular graph. □

Suppose G induces a star on $\{v_0,v_1,v_2,v_3\}$ centered at v_0. Let L_i be the set of vertices at distance i from v_0 in G. From Claim 2, we know that $L_1=\{v_1,v_2,v_3\}$, and moreover every $w\in L_2$ is adjacent to exactly two among v_1,v_2,v_3. Because $|G|\ge 6$, it must be the case that $L_2\ne\varnothing$. We break the rest of the proof into two cases.

Case $|L_2|=1$. Suppose $L_2=\{w\}$. By Claim 2, without loss of generality, w is adjacent to v_1 and v_2. Because $|G|\ge 6$, it must be the case that $L_3\ne\varnothing$. Take any $w'\in L_3$. Note that G induces a star on $\{w,v_1,v_2,w'\}$ centered at w. By Claim 2, $L_3=\{w'\}$ and $L_4=\varnothing$, which implies $|G|=6$. By Claim 5, G is a 3-regular graph, which is a contradiction.

Case $|L_2|\ge 2$. For every two $w_1,w_2\in L_2$, we claim that they do not have the same pairs of neighbors in L_1. Indeed, suppose on the contrary that both w_1 and w_2 are, without loss of generality, adjacent to v_1 and v_2 in L_1. Since v_2 is adjacent to

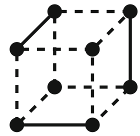


Fig. 4 $H_3^{\pm}$

v_0, w_1, w_2, by Claim 2, G does not induce a star on $\{v_0, v_1, w_1, w_2\}$ centered at v_1, and so $w_1w_2 \in E(G)$. Now we have two triangles $w_1w_2v_1$ and $w_1w_2v_2$, which by Claim 1 all have negative edges. Thus v_1 and v_2 are in the same part of the valid 3-coloring. Let $H^{\pm} := G^{\pm}[v_0, v_1, v_2, w_1]$. Then $\chi(H^{\pm}) \le 2$ and $H^{\pm}$ is a signed 4-cycle. Thus $\lambda_1(H^{\pm}) = \lambda_1(C_4) = 2$, where C_4 denotes the 4-cycle, contradicting $\lambda_1(H^{\pm}) \le \sqrt{3}$.

Assume for a moment that $|G| = 6$. In this subcase, $|L_2| = 2$ and $L_3 = \varnothing$, and so the degree of every vertex in L_2 is 2. By Claim 5, G is a 3-regular graph, which is a contradiction. Hereafter $|G| \ge 8$.

Because no two vertices in L_2 have the same pairs of neighbors in L_1, $|L_2| \le \binom{3}{2} = 3$. Because $|G| \ge 8$, it must be the case that $L_3 \ne \varnothing$. Take $w_1 \in L_2$ and $w' \in L_3$ such that $w_1w' \in E(G)$. Without loss of generality, suppose that w_1 is adjacent to v_1 and v_2. Since G induces a star on $\{v_1, v_2, w_1, w'\}$ centered at w_1, by Claim 2, w' is the only neighbor of w_1 in L_3, and w' has no neighbor in L_4. Now take an arbitrary vertex $w_2 \in L_2 \backslash \{w_1\}$. Since w_1 and w_2 do not have the same pairs of neighbors in L_1, the vertex w_2 is adjacent to only one of v_1 and v_2, and so $w_2w' \in E(G)$ by Claim 2. We can apply the previous argument to w_2 in place of w_1, and conclude that w' is the only neighbor of w_2 in L_3. Since $w_2 \in L_1 \backslash \{w_1\}$ was chosen arbitrarily, we know that $L_3 = \{w'\}$ and $L_4 = \varnothing$, which implies $|L_2| = 3$ and $|G| = 8$.

Since G is a 3-regular graph by Claim 5, it is easy to see that G must be the cubical graph. In view of Claim 5, $G^{\pm}$ is a signed cube that satisfies $A^2_{G^{\pm}} = 3I$, which means that every square of $G^{\pm}$ contains odd number of negative edges. Because $\chi(G^{\pm}) \le 3$, $G^{\pm}$ has no cycle with exactly one negative edge, and in particular every square of $G^{\pm}$ contains exactly one positive edge. At this point, it is not hard to deduce that $G^{\pm}$ is exactly $H_3^{\pm}$ in Fig. 4. However $\chi(H_3^{\pm}) = 4$, which is a contradiction. □

Proof of Theorem 1.12(c) Theorems 5.3 and 6.1 give

$$N_{\alpha,\beta}(d) \le d + 3N_{\alpha,\beta}(d)/7 + O_{\alpha,\beta}(1),$$

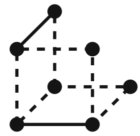
which implies

$$N_{\alpha,\beta}(d) \le 7d/4 + O_{\alpha,\beta}(1). \tag{10}$$

Comparing with Proposition 2.2, we get

$$\frac{k_3(\sqrt{3})d}{k_3(\sqrt{3}) - 1} - o(d) \le \frac{7d}{4} + O_{\alpha,\beta}(1),$$

Fig. 5 $\hat{H}_3^{\pm}$



which implies that $k_3(\sqrt{3}) \geq 7/3$. One can check that the signed graph $H_3^{\pm}$ in Fig. 4 satisfies

$$A^2_{H_3^{\pm}} = 3I,$$

and so $\mathrm{mult}(\sqrt{3}, H_3^{\pm}) = \mathrm{mult}(-\sqrt{3}, H_3^{\pm}) = 4$. By the Cauchy interlacing theorem, the signed graph $\hat{H}_3^{\pm}$ in Fig. 5, which is an induced subgraph of $H_3^{\pm}$ on 7 vertices, satisfies $\mathrm{mult}(\sqrt{3}, \hat{H}_3^{\pm}) = \mathrm{mult}(-\sqrt{3}, \hat{H}_3^{\pm}) = 3$. Moreover $\chi(\hat{H}_3^{\pm}) = 3$. Therefore $7/3$ can be achieved for $k_3(\sqrt{3})$ by $\hat{H}_3^{\pm}$. Now $k_3(\sqrt{3}) = 7/3$, and Proposition 2.2 provides a lower bound that matches (10) up to an additive constant. □

7 Algebraic Degree Argument

We use the following simple observation to derive the asymptotic formula of $N_{\alpha,\beta}(d)$ in the $k_p(\lambda) = \deg(\lambda)$ case, where $\deg(\lambda)$ denotes the algebraic degree of λ. In particular, the results in this section confirm Conjecture 1.11 when $\lambda \in \{\sqrt{2}, \sqrt{3}\}$ and $p \geq \lambda^2 + 1$.

Proposition 7.1 *For every algebraic integer $\lambda > 0$ and every signed graph $G^{\pm}$,*

$$\mathrm{mult}(\lambda, G^{\pm}) \leq \left|G^{\pm}\right| / \deg(\lambda).$$

In particular, $k_p(\lambda) \geq \deg(\lambda)$ for all $p \in \mathbb{N}$.

Proof If λ is an eigenvalue of a signed graph $G^{\pm}$ then each of its conjugates must also appear with equal multiplicity as eigenvalues of $G^{\pm}$. Hence $\mathrm{mult}(\lambda, G^{\pm}) \deg(\lambda) \leq \left|G^{\pm}\right|$. □

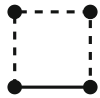
Proposition 7.2 *For $-1 \leq \beta < 0 \leq \alpha < 1$, set $\lambda = (1-\alpha)/(\alpha-\beta)$ and $p = \lfloor -\alpha/\beta \rfloor + 1$. If λ is an algebraic integer of degree at least 2, then*

$$N_{\alpha,\beta}(d) \leq \frac{\deg(\lambda)(d+1)}{\deg(\lambda)-1}.$$

If in addition $k_p(\lambda) = \deg(\lambda)$ and is achievable, then

$$N_{\alpha,\beta}(d) = \frac{\deg(\lambda) d}{\deg(\lambda)-1} + O_{\alpha,\beta}(1).$$

Fig. 6 $H_2^{\pm}$



Proof By Lemma 2.1, we see that if G is the graph associated to a spherical $\{\alpha, \beta\}$-code of size N in $\mathbb{R}^d$, then, setting $\mu = \alpha/(\alpha - \beta)$ as in Lemma 2.1, we have

$$\begin{aligned} d &\geq \operatorname{rank}(\lambda I - A_G + \mu J) \geq \operatorname{rank}(\lambda I - A_G) - 1 \\ &= N - \operatorname{mult}(\lambda, G) - 1 \geq \left(1 - \frac{1}{\deg(\lambda)}\right) N - 1, \end{aligned}$$

where the final step applies Proposition 7.1. This yields the first claim. If in addition $k_p(\lambda) = \deg(\lambda)$ and is achievable, then Proposition 2.2 gives a matching lower bound. □

Let us consider the case when λ is an algebraic integer of degree 2. Furthermore suppose that $k_p(\lambda) = 2$ and can be achieved by a signed graph $G^{\pm}$. Note that both λ and its conjugate element λ' must have multiplicity $\left|G^{\pm}\right|/2$ as the eigenvalues of $G^{\pm}$. Because the trace of $A_{G^{\pm}}$ is 0, we know that $\lambda + \lambda' = 0$. Therefore $\lambda = \sqrt{n}$ for some $n \in \mathbb{N}$ and $A_{G^{\pm}}^2 = nI$. It is natural to consider a signed n-dimensional hypercube $H_n^{\pm}$ used by Huang's recent spectacular proof of the sensitivity conjecture [7, Lemma 2.2], in which every square of $H_n^{\pm}$ contains 1 or 3 positive edges.

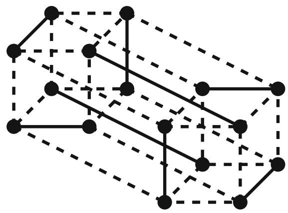
Proof of Theorem 1.12(d) For $\lambda \in \{\sqrt{2}, \sqrt{3}\}$ and $p \geq \lambda^2 + 1$, from Proposition 7.1 we know $k_p(\lambda) \geq 2$. In view of Proposition 7.2 it suffices to prove that 2 can be achieved for $k_p(\lambda)$. Consider the signed square $H_2^{\pm}$ in Fig. 6 and the signed cube $H_3^{\pm}$ in Fig. 4. In either signed graph, every square contains one positive edge and three negative edges. As a consequence

$$A_{H_n^{\pm}}^2 = nI, \quad \text{for } n = 2 \text{ and } 3,$$

which implies that the largest eigenvalue of $H_n^{\pm}$ is $\sqrt{n}$ with multiplicity 2^{n-1}. It is easy to check that $\chi(H_2^{\pm}) = 3$ and $\chi(H_3^{\pm}) = 4$. Thus $k_p(\sqrt{2}) = 2$ for $p \geq 3$, $k_p(\sqrt{3}) = 2$ for $p \geq 4$, and all of them are achievable. □

Remark The constructions $H_2^{\pm}$ and $H_3^{\pm}$ in Figs. 4 and 6 do not generalize for $\lambda = \sqrt{n}$ with $n \geq 5$ due to the additional constraint on the chromatic number. Suppose that $H^{\pm}$ is a signed n-dimensional hypercube such that $A_{H^{\pm}}^2 = nI$ and $\chi(H^{\pm}) < \infty$. Because $A_{H^{\pm}}^2 = nI$, every square of $H^{\pm}$ contains odd number of negative edges. Because $\chi(H^{\pm}) < \infty$, $H^{\pm}$ has no cycle with exactly one negative edge, and in particular every square of $H^{\pm}$ contains exactly one positive edge. Unfortunately, this puts a great restriction on n. On the one hand, because every positive edge is contained in $n-1$ squares, and each of the $2^{n-2}\binom{n}{2}$ squares in $H^{\pm}$ contains a positive edge, the number of positive edges is at least $2^{n-2}\binom{n}{2}/(n-1) = n2^{n-3}$. On the other hand, because the positive edges form a matching, there are at most 2^{n-1} of them.

Fig. 7 $H_4^{\pm}$



Therefore $n2^{n-3} \leq 2^{n-1}$ and so $n \leq 4$. In fact, in addition to $H_2^{\pm}$ and $H_3^{\pm}$, the signed 4-dimensional hypercube $H_4^{\pm}$ in Fig. 7 satisfies $A^2_{H_4^{\pm}} = 4I$ and $\chi(H_4^{\pm}) = 4$.

When $k(\lambda) = \deg(\lambda)$, the next result determines $k_p(\lambda)$ for all $p \in \mathbb{N}$. One can then derive the corresponding $N_{\alpha,\beta}(d)$ from Proposition 7.2. Note that $k(\lambda) = \deg(\lambda)$ if and only if there exists a graph with spectral radius λ whose characteristic polynomial is irreducible. A result of Mowshowitz [13] states that such a graph must be asymmetric[2]. Asymmetric graphs have at least 6 vertices. There are 8 such graphs on 6 vertices [5]. Among these 8 asymmetric graphs on 6 vertices, exactly 7 of them have irreducible characteristic polynomials,[3] hence their spectral radii satisfy $k(\lambda) = \deg(\lambda)$.

Proposition 7.3 *If λ is an algebraic integer and $k(\lambda) = \deg(\lambda)$, then $k_p(\lambda) = \deg(\lambda)$ and is achievable for all $p \in \mathbb{N}$.*

Proof Clearly $k_p(\lambda) \leq k_1(\lambda) = k(\lambda)$. Together with Proposition 7.1, we know that $\deg(\lambda) \leq k_p(\lambda) \leq k(\lambda)$. Thus if $k(\lambda) = \deg(\lambda)$, then $\deg(\lambda) = k_p(\lambda) = k(\lambda)$, and furthermore $k(\lambda)$ can be achieved for $k_p(\lambda)$ by the smallest graph whose spectral radius is exactly λ. □

Corollary 7.4 *For $-1 \leq \beta < 0 \leq \alpha < 1$, set $\lambda = (1-\alpha)/(\alpha-\beta)$ and $p = \lfloor -\alpha/\beta \rfloor + 1$. If λ is an algebraic integer and $k(\lambda) = \deg(\lambda)$, then*

$$N_{\alpha,\beta}(d) = \frac{\deg(\lambda)d}{\deg(\lambda)-1} + O_{\alpha,\beta}(1).$$ □

8 Signed Graphs with Large Eigenvalue Multiplicities

In contrast to Theorem 1.5, there exist connected signed graphs with bounded maximum degree and chromatic number and linear largest eigenvalue multiplicity. In this section, we show two such constructions. These constructions illustrate an important obstacle to proving Conjecture 1.11 following the current framework introduced in [10].

[2] An *asymmetric graph* is a graph for which there are no automorphisms other than the trivial one.

[3] It was asserted in [9, Sect. 4] that all 8 asymmetric graphs on 6 vertices have irreducible characteristic polynomials. However the characteristic polynomial of the asymmetric graph  is $x(x^5 - 8x^3 - 6x^2 + 8x + 6)$.

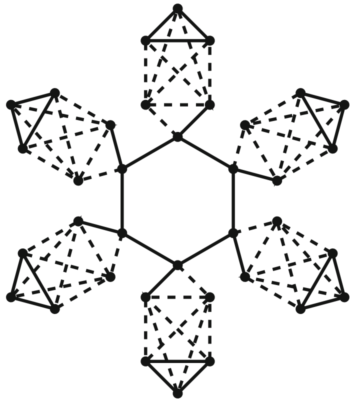


Fig. 8 $G_6^{\pm}$

Example 8.1 Let $n \geq 3$. Let $G_n^{\pm}$ be the signed graph consisting of (see Fig. 8 for an illustration of $G_6^{\pm}$)

(1) a positive n-cycle on $v_1, v_2, \ldots, v_n$,
(2) n copies of a signed K_5 with 3 positive edges forming a K_3, and
(3) for each $i \in \{1, \ldots, n\}$, a positive edge connecting v_i and u_i^+, a negative edge connecting v_i and u_i^-, where u_i^+ and u_i^- are the two vertices outside the positive K_3 in the i-th copy of K_5.

So $G_n^{\pm}$ is a signed graph on $6n$ vertices of maximum degree 5 and chromatic number 3. However the multiplicity of its largest eigenvalue is linear in $\left|G_n^{\pm}\right|$. Theorem 1.14 is an immediate consequence of the following result.

Proposition 8.2 *The largest eigenvalue of $G_n^{\pm}$ is $(\sqrt{33}+1)/2$ with multiplicity n.*

Proof We denote by $K_5^{\pm}$ the signed K_5 with 3 positive edges forming a K_3, and we compute the spectrum of $K_5^{\pm}$ to be $(1-\sqrt{33})/2, -1, -1, 1, (1+\sqrt{33})/2$. Because the largest eigenvalue $(\sqrt{33}+1)/2$ is simple, by symmetry the corresponding eigenvector assigns the same value to u_i^+ and u_i^-. For the i-th copy of $K_5^{\pm}$ in $G_n^{\pm}$, we can extend its top eigenvector to a vector $\boldsymbol{x_i}$ on $V(G_n^{\pm})$ by padding zeros. Since.

$$(A\boldsymbol{x_i})_{v_i} = (\boldsymbol{x_i})_{u_i^+} - (\boldsymbol{x_i})_{u_i^-} = 0,$$

where A denotes the signed adjacency matrix of $G_n^{\pm}$, the vector $\boldsymbol{x_i}$ is also an eigenvector of $G_n^{\pm}$ associated with the eigenvalue $(\sqrt{33}+1)/2$.

For every vector $\boldsymbol{x} \in \mathbb{R}^{V(G_n^{\pm})}$ that is perpendicular to all $\boldsymbol{x_i}$, $1 \leq i \leq n$, we claim that $\boldsymbol{x}^\intercal A\boldsymbol{x} \leq 3\boldsymbol{x}^\intercal\boldsymbol{x}$, and so all the eigenvalues other than the ones corresponding to $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ are at most 3. Take such a vector $\boldsymbol{x}$, and set $U = \{u_1^+, u_1^-, \ldots, u_n^+, u_n^-\}$ and $V = \{v_1, \ldots, v_n\}$. We take the orthogonal decomposition $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$ such that $\boldsymbol{y}$ and $\boldsymbol{z}$ are supported respectively on $V(G_n^{\pm})\backslash V$ and $U \cup V$. In particular, for every $i \in \{1, \ldots, n\}$,

$$(\boldsymbol{y})_{u_i^+} = (\boldsymbol{y})_{u_i^-} = \tfrac{1}{2}\left(\boldsymbol{x}_{u_i^+} + \boldsymbol{x}_{u_i^-}\right) \quad \text{and} \quad (\boldsymbol{z})_{u_i^+} = -(\boldsymbol{z})_{u_i^-} = \tfrac{1}{2}\left(\boldsymbol{x}_{u_i^+} - \boldsymbol{x}_{u_i^-}\right).$$

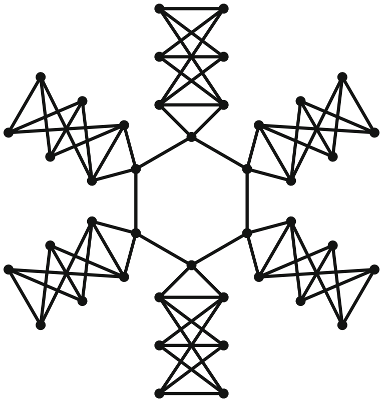


Fig. 9 H_6

One can check that $\boldsymbol{y}^\intercal A\boldsymbol{z} = 0$. We can simplify

$$\boldsymbol{x}^\intercal A\boldsymbol{x} = (\boldsymbol{y}+\boldsymbol{z})^\intercal A(\boldsymbol{y}+\boldsymbol{z}) = \boldsymbol{y}^\intercal A\boldsymbol{y} + \boldsymbol{z}^\intercal A\boldsymbol{z}.$$

Since $\boldsymbol{x}$ and $\boldsymbol{z}$ are both orthogonal to each $\boldsymbol{x_i}$, so is $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{z}$. By the Courant–Fischer theorem, we obtain $\boldsymbol{y}^\intercal A\boldsymbol{y} \le \lambda_2(K_5^{\pm})\boldsymbol{y}^\intercal\boldsymbol{y} = \boldsymbol{y}^\intercal\boldsymbol{y}$. As $\boldsymbol{z}$ is supported on $U \cup V$, we bound $\boldsymbol{z}^\intercal A\boldsymbol{z}$ by bounding the spectral radius of $G_n^{\pm}[U \cup V]$. Since the chromatic number of $G_n^{\pm}[U \cup V]$ is 2, the induced signed subgraph shares the same spectral radius with its underlying graph, denoted H, on $U \cup V$. Notice that the vector that assigns 1 to U and 2 to V is an eigenvector of H with positive components associated with the eigenvalue 3. By the Perron–Frobenius theorem, the spectral radius of H is 3. Thus $\boldsymbol{z}^\intercal A\boldsymbol{z} \le 3\boldsymbol{z}^\intercal\boldsymbol{z}$. Recall that $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$ is an orthogonal decomposition. Thus

$$\boldsymbol{x}A\boldsymbol{x} = \boldsymbol{y}^\intercal A\boldsymbol{y} + \boldsymbol{z}^\intercal A\boldsymbol{z} \le \boldsymbol{y}^\intercal\boldsymbol{y} + 3\boldsymbol{z}^\intercal\boldsymbol{z} \le 3(\boldsymbol{y}^\intercal\boldsymbol{y} + \boldsymbol{z}^\intercal\boldsymbol{z}) = 3\boldsymbol{x}^\intercal\boldsymbol{x}. \qquad \square$$

Even if we restrict the signed graph $G^{\pm}$ to be all-negative, its largest eigenvalue multiplicity could still be linear in $\left|G^{\pm}\right|$. It suffices to construct the underlying graph G with bounded maximum degree whose smallest eigenvalue multiplicity is linear in $|G|$.

Example 8.3 Let $n \ge 3$. Let H_n be the (unsigned) graph consisting of (see Fig. 9 for an illustration of H_6)

(1) an n-cycle on $v_1, v_2, \ldots, v_n$,
(2) n copies of $K_{3,3}$, and
(3) for each $i \in \{1, \ldots, n\}$, two edges connecting v_i to u_i^1 and u_i^2, where u_i^1 and u_i^2 are two adjacent vertices in the i-th copy of $K_{3,3}$.

So H_n is a graph on $7n$ vertices of maximum degree 4. Moreover, since the chromatic number of H_n is 3, the corresponding all-negative signed graph has the same chromatic number.

Proposition 8.4 *The smallest eigenvalue of H_n is -3 with multiplicity n.*

Proof We compute the spectrum of $K_{3,3}$ to be $3, 0, 0, 0, 0, -3$. For the i-th copy of $K_{3,3}$, we can extend the eigenvector associated with its smallest eigenvalue -3 to an eigenvector $\boldsymbol{x_i}$ on $V(H_n)$ by padding zeros. To prove that all the eigenvalues other than the ones corresponding to $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ are at least $-(1+\sqrt{3})$, it suffices to show that $\boldsymbol{x}^\intercal A\boldsymbol{x} \geq -(1+\sqrt{3})\boldsymbol{x}^\intercal\boldsymbol{x}$ for every vector $\boldsymbol{x} \in \mathbb{R}^{V(G_n^{\pm})}$ that is perpendicular to all $\boldsymbol{x_i}$, $1 \leq i \leq n$. Take such a vector $\boldsymbol{x}$ and take the orthogonal decomposition $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$ such that $\boldsymbol{y}$ and $\boldsymbol{z}$ are supported respectively on $V(H_n)\setminus V$ and V, where $V = \{v_1, \ldots, v_n\}$. Because $\boldsymbol{x}$ and $\boldsymbol{z}$ are orthogonal to each $\boldsymbol{x_i}$, so is $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{z}$. By the Courant–Fischer theorem, we obtain $\boldsymbol{y}^\intercal A\boldsymbol{y} \geq \lambda_5(K_{3,3})\boldsymbol{y}^\intercal\boldsymbol{y} = 0$. We can simplify

$$\boldsymbol{x}^\intercal A\boldsymbol{x} = (\boldsymbol{y}+\boldsymbol{z})^\intercal A(\boldsymbol{y}+\boldsymbol{z}) \geq 2\boldsymbol{y}^\intercal A\boldsymbol{z} + \boldsymbol{z}^\intercal A\boldsymbol{z}, \tag{11}$$

where A denotes the adjacency matrix of H_n. Let $\bar{H}$ be the connected graph consisting of the n-cycle on $v_1, \ldots, v_n$ and two edges connecting v_i to u_i^1 and u_i^2 for each $i \in \{1, \ldots, n\}$. Let $\bar{\boldsymbol{x}}$ be the restriction of $\boldsymbol{x}$ on $V(\bar{H})$. Then the right hand side of (11) is equal to $\bar{\boldsymbol{x}}^\intercal \bar{A}\bar{\boldsymbol{x}}$, where $\bar{A}$ denotes the adjacency matrix of $\bar{H}$. Notice that the vector that assigns $1+\sqrt{3}$ to v_i and 1 to both u_i^1 and u_i^2 for every $i \in \{1, \ldots, n\}$ is an eigenvector of $\bar{H}_n$ with positive components associated with the eigenvalue $1+\sqrt{3}$. By the Perron–Frobenius theorem, the spectral radius of $\bar{H}$ is $1+\sqrt{3}$. Thus

$$\boldsymbol{x}^\intercal A\boldsymbol{x} \geq \bar{\boldsymbol{x}}^\intercal \bar{A}\bar{\boldsymbol{x}} \geq -(1+\sqrt{3})\bar{\boldsymbol{x}}^\intercal\bar{\boldsymbol{x}} \geq -(1+\sqrt{3})\boldsymbol{x}^\intercal\boldsymbol{x}. \qquad \square$$

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